Some Topics on Dirichlet Forms and Non-symmetric Markov Processes

Jing Zhang

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By: Jing Zhang

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Dr. Xiaowen Zhou ______ Examiner

Dr. Arusharka Sen ______ Examiner

Dr. Wei Sun ______ Thesis Supervisor

Approved by

____

Chair of Department or Graduate Program Director

April 1, 2016 ______________________________

Dean of Faculty

ABSTRACT

Some topics on Dirichlet forms and non-symmetric Markov processes

Jing Zhang, Ph.D.

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In this thesis, we discuss three topics on Dirichlet forms and non-symmetric Markov processes.

First, we explore the analytic structure of non-symmetric Markov processes. Let U be an open set of $\mathbf{R}^n$, m a positive Radon measure on U, and $(P_t)_{t>0}$ a strongly continuous contraction sub-Markovian semigroup on $L^2(U;m)$. We give an explicit Lévy-Khintchine type representation of the generator A of $(P_t)_{t>0}$. If $(P_t)_{t>0}$ is an analytic semigroup, we give an explicit characterization of the semi-Dirichlet form $\mathcal{E}$ associated with $(P_t)_{t>0}$.

Second, we consider the Dirichlet boundary value problem for a general class of second order non-symmetric elliptic operators L with singular coefficients. We show that there exists a unique, bounded continuous solution by using the theory of Dirichlet forms and heat kernel estimates. Also, we give a probabilistic representation of the non-symmetric semigroup generated by L.

Finally, we present new results on Hunt's hypothesis (H) for Lévy processes. These include a comparison result on Lévy processes which implies that big jumps have no effect on the validity of (H), a new necessary and sufficient condition for (H), and an extended Kanda-Forst-Rao theorem.

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Contribution of Authors

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Both authors contributed equally to this work.

Chapter 3 of the thesis is based on the following joint paper of Chuanzhong Chen, Wei Sun and Jing Zhang:

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All authors contributed equally to this work.

Chapter 4 of the thesis is based on the following joint paper of Zechun Hu, Wei Sun and Jing Zhang:

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All authors contributed equally to this work.

Contents

Chapter 1

Introduction

Dirichlet forms have been used to study Markov processes since the fundamental work of Fukushima and Silverstein (see Fukushima (1971) and Silverstein (1974)). Two of their advantages are in handling Markov processes with singular coefficients and Markov processes with infinite dimensional state spaces.

Historically, the connection between Dirichlet forms and Markov processes was initially established between symmetric Dirichlet forms and time-reversal Markov processes, and then extended to include non-symmetric Markov processes. In the following years, many important results on symmetric Dirichlet forms were generalized to non-symmetric Dirichlet forms or, more generally, semi-Dirichlet forms. In this thesis, we will further explore the connection between Markov processes and non-symmetric Dirichlet forms.

Let $(X_t)_{t\geq 0}$ be a general right continuous Markov process on $\mathbf{R}^n$ or, more generally, on an open set U of $\mathbf{R}^n$. In the first part of the thesis (Chapter 2), we discuss the analytic structure of $(X_t)_{t\geq 0}$. Denote by $(P_t)_{t>0}$ the transition semigroup of $(X_t)_{t\geq 0}$. Suppose that there is a positive Radon measure m on U such that $(P_t)_{t>0}$ acts as a strongly continuous contraction semigroup on $L^2(U;m)$. We will give an explicit Lévy-Khintchine type representation of the L^2-generator A of $(P_t)_{t>0}$. This result generalizes the classical Courrège representation theorem for generators of Feller processes.

If the diffusion part of $(X_t)_{t\geq 0}$ corresponds to a differential operator with very singular coefficients, we adopt the framework of semi-Dirichlet forms to investigate the analytic structure of $(X_t)_{t\geq 0}$ and we will present a Lévy-Khintchine type representation of semi-Dirichlet forms, which generalizes the classical Beurling-Deny formula of symmetric Dirichlet forms on open sets of $\mathbf{R}^n$.

Further, we apply some ideas of deriving the above Lévy-Khintchine type representations to characterize the structure of general regular semi-Dirichlet forms. Recently, there is new interest in further developing the theory of semi-Dirichlet forms. For example, semi-Dirichlet forms are used to construct and study Hunt processes with jumps (cf. Fukushima and Uemura (2012), Uemura (2014a), Uemura (2014b), Schilling and Wang (2015)), the stochastic calculus of nearly-symmetric Markov processes has been generalized to the semi-Dirichlet forms setting (cf. Ma et al. (2012), Oshima (2013), Chen et al. (2014), Ma et al. (2014)). Our characterization of the structure of semi-Dirichlet forms can help people better understand properties of their associated non-symmetric Markov processes and may have potential applications.

In the second part of the thesis (Chapter 3), we use the theory of non-symmetric Dirichlet forms to consider the Dirichlet boundary value problem. We use a probabilistic approach to show that there exists a unique, bounded continuous solution to the Dirichlet boundary value problem for a general class of second order non-symmetric elliptic operators L with singular coefficients, which does not necessary have the maximum principle. Our conditions are even weaker than the ordinary conditions assumed by virtue of the PDE methods. By using some similar techniques, we also give a probabilistic representation of the non-symmetric semigroups generated by L. The obtained result generalizes the previous result of Lunt et al. (1998) from the case of symmetric diffusion matrix A to the non-symmetric case.

Note that the sector condition plays a crucial role in the theory of non-symmetric Dirichlet forms. In particular, it implies that any Markov process associated with a non-symmetric Dirichlet form or a semi-Dirichlet form must satisfy Hunt's hypothesis (H) (cf. Silverstein (1977), Fitzsimmons (2001)). To further extend the theory of Dirichlet forms to handle more general non-symmetric Markov processes, we are interested in the question which Markov processes satisfy (H). This is a long-standing open problem and is even not completely solved for Lévy processes. In the third part of the thesis (Chapter 4), we discuss (H) for Lévy processes.

We will present a comparison result on Lévy processes which shows that big jumps have no effect on the validity of (H) in some sense. Based on this result and the Kanda-Forst-Rao theorem, we give examples of subordinators satisfying (H). Afterwards, we give a new necessary and sufficient condition for (H) and obtain an extended Kanda-Forst-Rao theorem. By virtue of this theorem, we give a new class of Lévy processes satisfying (H). Finally, we construct a type of subordinators that does not satisfy Rao's condition.

1.1 Preliminaries

In this section, we will give a brief introduction to semi-Dirichlet forms and review some known results related to the Dirichlet boundary value problem and Hunt's hypothesis (H) for Markov processes.

1.1.1 A Dirichlet form primer

The materials of this subsection are taken from Ma and Röckner (1992) and Ma et al. (2015). We refer the readers to the monographs Ma and Röckner (1992), Fukushima et al. (2011) and Oshima (2013) for complete descriptions of the theory of Dirichlet forms and semi-Dirichlet forms.

Let E be a Hausdorff topological space and m a σ-finite positive measure on its Borel σ-algebra $\mathcal{B}(E)$. For a bilinear form $\mathcal{E}$ with domain $D(\mathcal{E}) \subset L^2(E;m)$, we define for $u, v \in D(\mathcal{E})$,

$$
\begin{aligned}
\tilde{\mathcal{E}}(u,v) &:= \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u)),\\
\check{\mathcal{E}}(u,v) &:= \frac{1}{2}(\mathcal{E}(u,v) - \mathcal{E}(v,u)),\\
\hat{\mathcal{E}}(u,v) &:= \mathcal{E}(v,u).
\end{aligned}
$$

$\tilde{\mathcal{E}}$ is called the *symmetric part* of $\mathcal{E}$, $\check{\mathcal{E}}$ the *antisymmetric part* of $\mathcal{E}$ and $\hat{\mathcal{E}}$ the *dual form* of $\mathcal{E}$.

For $\alpha > 0$, we write

$$\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v), \quad \forall u,v \in D(\mathcal{E}).$$

Hereafter, $(\cdot,\cdot)$ denotes the usual inner product of $L^2(E;m)$. For $u,v : E \to \mathbf{R}$, we set $u \vee v := \sup(u,v)$, $u \wedge v := \inf(u,v)$, $u^+ := u \vee 0$.

Definition 1.1 *Let $(\mathcal{E}, D(\mathcal{E}))$ be a bilinear form on $L^2(E;m)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is called a semi-Dirichlet form if the following three conditions are satisfied.*

(a) $D(\mathcal{E})$ is dense in $L^2(E;m)$, and the symmetric part $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is positive definite and closed on $L^2(E;m)$.

(b) (weak sector condition) There exists $K > 0$ (called continuity constant) such that

$$|\mathcal{E}_1(u,v)| \le K\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}, \quad \forall u,v \in D(\mathcal{E}).$$

(c) (semi-Dirichlet property) If $u \in D(\mathcal{E})$, then

$$u^+ \wedge 1 \in D(\mathcal{E}) \text{ and } \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \ge 0.$$

Remark 1.2 *(i) $(\mathcal{E}, D(\mathcal{E}))$ is called a Dirichlet form if both $\mathcal{E}$ and $\hat{\mathcal{E}}$ are semi-Dirichlet forms.*

(ii) Let $(T_t)_{t\ge 0}$ and $(G_\alpha)_{\alpha\ge 0}$ be the semigroup and the resolvent associated with $(\mathcal{E}, D(\mathcal{E}))$, respectively. Then the semi-Dirichlet property is equivalent to the sub-Markov property for $(T_t)_{t\ge 0}$ or for $(G_\alpha)_{\alpha\ge 0}$, respectively. The sub-Markov property of $(T_t)_{t\ge 0}$ and $(G_\alpha)_{\alpha\ge 0}$ is defined as:

$$f \in L^2(E;m), 0 \le f \le 1 \ m\text{-a.e.} \Rightarrow 0 \le T_t f \le 1 \ m\text{-a.e.}, \ \forall t > 0.$$

and

$$f \in L^2(E;m), 0 \le f \le 1 \ m\text{-a.e.} \Rightarrow 0 \le \alpha G_\alpha f \le 1 \ m\text{-a.e.}, \ \forall \alpha > 0.$$

Definition 1.3 *A semi-Dirichlet form on $L^2(E;m)$ is called regular if $C_0(E) \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm and $C_0(E) \cap D(\mathcal{E})$ is dense in $C_0(E)$ w.r.t. the uniform norm $\|\cdot\|_\infty$. Hereafter, $C_0(E)$ denotes the space of all real continuous functions on E with compact support.*

We adjoin an extra point Δ (the cemetery) to E and write $E_\Delta := E \cup \{\Delta\}$. Any function $f : E \to \mathbf{R}$ is considered as a function on E_Δ by putting $f(\Delta) = 0$. Let $(\Omega, \mathcal{M})$ be a measurable space and $(X_t)_{t\geq 0}$ be a stochastic process with state space E and life time ζ on $(\Omega, \mathcal{M})$. Here ζ is called a *life time* if for all $\omega \in \Omega, X_t(\omega) \in E$ whenever $t < \zeta(\omega)$ and $X_t(\omega) = \Delta$ for all $t \geq \zeta(\omega)$.

Definition 1.4 *A collection* $\mathbf{M} := (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ *is called a right process with state space* E *and life time* ζ *if it has the following properties:*

(i) There exists a filtration $(\mathcal{M}_t)$ *on* $(\Omega, \mathcal{M})$ *such that* $(X_t)_{t\geq 0}$ *is an* $(\mathcal{M}_t)$*-adapted stochastic process with state space* E *and life time* ζ*.*

(ii) For each $t \geq 0$*, there exists a shift operator* $\theta_t : \Omega \to \Omega$ *such that* $X_s \circ \theta_t = X_{s+t}$ *for all* $s, t \geq 0$*.*

(iii) P_x*,* $x \in E_\Delta$*, are probability measures on* $(\Omega, \mathcal{M})$ *such that* $t \to P_x(\Gamma)$ *is* $\mathcal{B}(E_\Delta)^*$*-measurable for each* $\Gamma \in \mathcal{M}$ *resp.* $\mathcal{B}(E_\Delta)$*-measurable if* $\Gamma \in \sigma\{X_s | s \in [0, \infty)\}$ *and* $P_\Delta(X_0 = \Delta) = 1$*. Here* $\mathcal{B}(E_\Delta)^*$ *denotes the* σ*-algebra consisting of all universally measurable subsets of* E_Δ*.*

(iv) (normal property) $P_x(X_0 = x) = 1$ *for all* $x \in E_\Delta$*.*

(v) (right continuity) For each $\omega \in \Omega$*,* $t \mapsto X_t(\omega)$ *is right continuous on* $[0, \infty)$*.*

(vi) (strong Markov property) $(\mathcal{M}_t)$ *is right continuous and for every* $(\mathcal{M}_t)$*-stopping time* σ *and every* $\mu \in \mathcal{P}(E_\Delta)$*,*

$$P_\mu(X_{\sigma+t} \in A | \mathcal{M}_\sigma) = P_{X_\sigma}(X_t \in A), \quad P_\mu\text{−a.s.}$$

for all $A \in \mathcal{B}(E_\Delta)$*,* $t \geq 0$*, where* $\mathcal{P}(E_\Delta)$ *denotes the family of all probability measures on* $(E_\Delta, \mathcal{B}(E_\Delta))$ *and*

$$\mathcal{M}_\sigma := \{\Gamma \in \mathcal{M} | \Gamma \cap \{\sigma \leq t\} \in \mathcal{M}_t \ \text{ for } \ \text{all } t \geq 0\}.$$

For a right process $\mathbf{M}$, we define the transition semigroup of $(X_t)_{t\geq 0}$ by

$$P_t f(x) := E_x[f(X_t)], \quad x \in E, t \geq 0, f \in \mathcal{B}(E)^+. \tag{1.1.1}$$

Hereafter, $\mathcal{B}(E)^+$ denotes the set of all positive Borel measurable functions on E and $\mathcal{B}_b(E)$ denotes the set of all bounded Borel measurable functions on E. $\mathbf{M}$ is said to be associated with a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ if and only if $P_t f$ is an m-version of $T_t f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E;m)$ and all $t > 0$.

The correspondence between Markov processes and Dirichlet forms was first established by Fukushima. He showed that any regular symmetric Dirichlet form is associated with a Hunt process (see Fukushima (1971)). In the 90s of the last century, Albeverio, Ma and Röckner developed the concept of quasi-regular Dirichlet forms.

Let $(\mathcal{E}, D(\mathcal{E}))$ be a semi-Dirichlet form on $L^2(E;m)$. Then there exists a right process $\mathbf{M}$ associated with $(\mathcal{E}, D(\mathcal{E}))$ if and only if $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular (see Ma et al. (1995)). We refer the readers to Ma et al. (1995) for the definition of quasi-regular semi-Dirichlet. It is known that any quasi-regular semi-Dirichlet form is quasi-homeomorphic to a regular semi-Dirichlet form (cf. Hu et al. (2006)). So the fruitful results for the regular case can be transferred to the quasi-regular case.

The Beurling-Deny formula plays a fundamental role in studying symmetric Dirichlet forms and their associated Markov processes. For a regular symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$, the Beurling-Deny formula tells us that $(\mathcal{E}, D(\mathcal{E}))$ can be expressed for $u, v \in C_0(E) \cap D(\mathcal{E})$ as:

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^c(u,v) + \int_{E\times E\setminus d} (u(x)-u(y))(v(x)-v(y))J(dxdy) \\ &\quad + \int_E u(x)v(x)K(dx). \qquad (1.1.2)\end{aligned}$$

Here d denotes the diagonal of $E \times E$, $\mathcal{E}^c(u,v)$ is a symmetric bilinear form with domain $D(\mathcal{E}^c) = C_0(E) \cap D(\mathcal{E})$ and satisfies the strong local property:

$$\mathcal{E}^c(u,v) = 0 \text{ for } u \in D(\mathcal{E}^c) \text{ and } v \in I(u),$$

where

$$I(u) := \{g \in D(\mathcal{E}^c) : g \text{ is constant on a neighbourhood of } \mathrm{supp}[u]\}.$$

J is a symmetric positive Radon measure on $E \times E\setminus d$ and K is a positive Radon measure on E. Such $\mathcal{E}^c$, J and K are uniquely determined by $\mathcal{E}$.

The Beurling-Deny formula of a symmetric Dirichlet form on an open set of $\mathbf{R}^n$ is as follows:

Theorem 1.5 *Let U be an open set of $\mathbf{R}^n$ and m be a positive Radon measure on U such that* $\operatorname{supp}[m] = U$. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a regular symmetric Dirichlet form on $L^2(U, m)$ such that $C_0^\infty(U) \subset D(\mathcal{E})$, where $C_0^\infty(U)$ is the space of infinitely differentiable functions with compact support in U. Then, $(\mathcal{E}, C_0^\infty(U))$ can be expressed uniquely for $u, v \in C_0^\infty(U)$ as follows:*

$$\begin{aligned}\mathcal{E}(u,v) &= \sum_{i,j=1}^{n} \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} d\nu_{ij} + \int_{U\times U\backslash d} (u(x)-u(y))(v(x)-v(y))J(dxdy) \\ &\quad + \int_U u(x)v(x)K(dx). \end{aligned} \tag{1.1.3}$$

For $1 \le i, j \le n$, ν_{ij} is a Radon measure on U such that for every $K \subset U$, K compact, $\nu_{ij}(K) = \nu_{ji}(K)$ and $\sum_{i,j=1}^n \xi_i\xi_j\nu_{ij}(K) \ge 0$ for all $\xi_1, \cdots, \xi_n \in \mathbf{R}^n$.

Furthermore, the structure of $\mathcal{E}^c$ is characterized by the local part of mutual energy measures. Let $u, v \in C_0(E) \cap D(\mathcal{E})$. Then, there exists a unique signed Radon measure $\mu^c_{<u,v>}$ on E such that

$$\int_E f d\mu^c_{<u,v>} = \mathcal{E}^c(uf, v) + \mathcal{E}^c(vf, u) - \mathcal{E}^c(uv, f), \quad f \in C_0(E) \cap D(\mathcal{E}).$$

We have $\mathcal{E}^c(u,v) = \frac{1}{2}\mu^c_{<u,v>}(E)$ and $\mu^c_{<u,v>}$ obeys LeJan's transformation rule:

$$d\mu^c_{<\Phi(u_1,\ldots,u_m),v>} = \sum_{i=1}^{m} \Phi_{x_i}(u_1,\ldots,u_m) d\mu^c_{<u_i,v>},$$

for any $\Phi \in C^1(\mathbf{R}^m)$ with $\Phi(0) = 0$ and $u_1, \ldots, u_m, v \in C_0(E) \cap D(\mathcal{E})$.

Proofs of the above structure results on symmetric Dirichlet forms can be found in Chapter 3 of Fukushima et al. (2011). When non-symmetric Dirichlet forms or, more generally, semi-Dirichlet forms are considered, things become complicated. Through introducing the SPV integrable condition, Hu et al. (2006) has generalized (1.1.2) to the semi-Dirichlet forms setting.

Definition 1.6 *Let Q be a σ-finite positive measure on $E \times E\backslash d$. A measurable function f on $E \times E\backslash d$ is said to be integrable with respect to Q in the sense of symmetric principle value (abbreviated by SPV integrable), if there exists an increasing sequence $\{A_n\}_{n\geq 1}$ of subsets of $E \times E\backslash d$ satisfying $Q((E \times E\backslash d)\backslash(\bigcup_{n\geq 1} A_n)) = 0$, $I_{A_n}(x, y) = I_{A_n}(y, x)$ for all $x, y \in E$, f is integrable on each A_n, $n \geq 1$, and for any sequence $\{A_n\}_{n\geq 1}$ with these properties, the limit*

$$SPV \int_{E\times E\backslash d} f(x, y)Q(dxdy) := \lim_{n\to\infty} \int_{A_n} f(x, y)Q(dxdy)$$

exists and is independent of the specific choice of the sequence $\{A_n\}_{n\geq 1}$.

Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2(E; m)$. Then, there exist a unique positive Radon measure J on $E \times E\backslash d$ and a unique positive Radon measure K on E such that for $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in I(v)$,

$$\mathcal{E}(u, v) = \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)J(dxdy) + \int_E u(x)v(x)K(dx).$$

Define $\mathcal{A}(v) := \{f \in C_0(E) \cap D(\mathcal{E}) : (f(y) - f(x))v(y) \text{ is SPV integrable w.r.t. } J\}$. Then, for $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in \mathcal{A}(v)$, we have the unique decomposition:

$$\begin{aligned}\mathcal{E}(u, v) \quad &= \quad \mathcal{E}^c(u, v) + SPV \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)J(dxdy) \\ &\quad + \int_E u(x)v(x)K(dx), \end{aligned} \tag{1.1.4}$$

where $\mathcal{E}^c(u, v)$ satisfies the *left strong local property* in the sense that $I(v) \subset \mathcal{A}(v)$ and $\mathcal{E}^c(u, v) = 0$ whenever $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in I(v)$. In general, the SPV integrable condition cannot be dropped for the decomposition (1.1.4) to hold (see Hu et al. (2010) for an example).

Hu et al. (2009, 2010) investigate the structure of non-symmetric Dirichlet forms and characterize their diffusion parts. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a regular (non-symmetric) Dirichlet form. Since the dual form $(\hat{\mathcal{E}}, D(\mathcal{E}))$ of $(\mathcal{E}, D(\mathcal{E}))$ also satisfies the semi-Dirichlet property, we have the decomposition:

$$\hat{\mathcal{E}}(u, v) \quad = \quad \hat{\mathcal{E}}^c(u, v) + SPV \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)\hat{J}(dxdy)$$

$$+\int_E u(x)v(x)\hat{K}(dx) \tag{1.1.5}$$

for $v \in C_0(E)\cap D(\mathcal{E})$ and $u \in \hat{\mathcal{A}}(v) := \{f \in C_0(E)\cap D(\mathcal{E}) : (f(y)-f(x))v(y)$ is SPV integrable w.r.t. $\hat{J}\}$. Note that $\hat{J}(dxdy) = J(dydx)$ and it can be shown that $\hat{\mathcal{A}}(v) = \mathcal{A}(v)$ for Dirichlet forms (cf. Hu et al. (2010)). Let $u, v \in C_0(E)\cap D(\mathcal{E})$ satisfying $(u(y)-u(x))v(y)$ is SPV integrable w.r.t. J. By (1.1.4) and (1.1.5), we get

$$\begin{aligned}
\check{\mathcal{E}}(u,v) &:= \frac{1}{2}(\mathcal{E}(u,v)-\mathcal{E}(v,u)) \\
&= \frac{1}{2}(\mathcal{E}^c(u,v)-\hat{\mathcal{E}}^c(u,v)) + SPV\int_{E\times E\backslash d} 2(u(y)-u(x))v(y)\frac{J-\hat{J}}{2}(dxdy) \\
&\quad + \int_E u(x)v(x)\frac{K-\hat{K}}{2}(dx).
\end{aligned}$$

Define

$$\check{\mathcal{E}}^c(u,v) := \frac{1}{2}(\mathcal{E}^c(u,v)-\hat{\mathcal{E}}^c(u,v))$$

and refer it as the *co-symmetric* diffusion part. Then, the diffusion part $\mathcal{E}^c$ is uniquely decomposed into the symmetric part and the co-symmetric part as follows:

$$\mathcal{E}^c(u,v) = \tilde{\mathcal{E}}^c(u,v) + \check{\mathcal{E}}^c(u,v).$$

Since $\tilde{\mathcal{E}}^c$ obeys LeJan's transformation rule, to understand the structure of $\mathcal{E}$, we need only concentrate on $\check{\mathcal{E}}^c$. In Hu et al. (2010), a LeJan type transformation rule is derived for $\check{\mathcal{E}}^c$ under the SPV integrable condition. This result has been used to study Markov processes associated with non-symmetric Dirichlet forms. For example, it plays a crucial role in investigating the strong continuity of generalized Feynman-Kac semigroups for nearly-symmetric Markov processes (cf. Ma and Sun (2012)).

1.1.2 Dirichlet boundary value problem

Using probabilistic approaches to solve boundary value problems has a long history. The pioneering work goes back to Kakutani (1944), who used Brownian motion to represent the solution of the classical Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D, \end{cases}$$

where Δ is the Laplacian operator, D is a bounded domain of $\mathbf{R}^n$, and f is a real-valued continuous function defined on the boundary ∂D of D.

Chen and Zhao (1995) used the Dirichley form theory to consider the following Dirichlet boundary value problem:

$$\begin{cases} Lu = 0 & \text{in } D \\ u = f & \text{on } \partial D, \end{cases} \tag{1.1.6}$$

with operator

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c, \tag{1.1.7}$$

where $A(x) = (a_{ij}(x))_{i,j=1}^{n}$ is a $n \times n$ symmetric positive definite matrix satisfying the uniform elliptic condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \ \text{ for any } \xi = (\xi_i)_{i=1}^{n} \in \mathbf{R}^n, x \in D \tag{1.1.8}$$

for some constant $0 < \lambda \leq 1$; $b = (b_1, \ldots, b_n)^*$ and c are Borel measurable functions on D such that

$$I_D|b|^2 \in K_n \quad \text{and} \quad I_D c \in K_n.$$

Hereafter, we use * to denote the transpose of a vector or matrix, use $|\cdot|$ and $\langle \cdot, \cdot \rangle$ to denote respectively the standard norm and inner product of $\mathbf{R}^n$, and use K_n to denote the Kato class measures. Recall that a real-valued measurable function f on $\mathbf{R}^n$ is said to be in K_n if and only if

$$\lim_{\alpha \downarrow 0}\left[\sup_x \int_{|y-x|\leq\alpha} |x-y|^{-(n-2)}|f(y)|dy\right] = 0, \quad \text{if} \quad n \geq 3,$$

$$\lim_{\alpha \downarrow 0}\left[\sup_x \int_{|y-x|\leq\alpha} (-\ln|x-y|)|f(y)|dy\right] = 0, \quad \text{if} \quad n = 2,$$

and

$$\sup_x \int_{|y-x|\leq 1} |f(y)|dy < \infty, \quad \text{if} \quad n = 1.$$

By using Hölder's inequality, one can show that $L^p(\mathbf{R}^n) \subset K_n$ for $p > n/2$. For any function f in K_n, and any $\varepsilon > 0$, there exists a constant $A(\varepsilon) > 0$ such that for any $u \in H^{1,2}(\mathbf{R}^n)$ (cf. Kato (1980))

$$\int_{\mathbf{R}^n} |f|u^2 dx \leq \varepsilon \int_{\mathbf{R}^n} |\nabla u|^2 dx + A(\varepsilon) \int_{\mathbf{R}^n} |u|^2 dx, \tag{1.1.9}$$

where $H^{1,2}(\mathbf{R}^n)$ is the Sobolev space on $\mathbf{R}^n$ with norm

$$\|f\|_{H^{1,2}} := \left(\int_{\mathbf{R}^n} |\nabla f(x)|^2 dx + \int_{\mathbf{R}^n} |f(x)|^2 dx \right)^{1/2}.$$

In (1.1.6), $Lu = 0$ in D is understood in the distributional sense:

$$u \in H^{1,2}(D) \text{ and } \mathcal{E}(u, \phi) = 0 \text{ for every } \phi \in C_0^\infty(D),$$

where $(\mathcal{E}, D(\mathcal{E}))$ is the bilinear form associated with the operator L given by (1.1.7):

$$\begin{aligned} \mathcal{E}(u,v) &= \frac{1}{2} \sum_{i,j=1}^n \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^n \int_D b_i(x) \frac{\partial u}{\partial x_i} v(x) dx \\ &\quad - \int_D c(x) u(x) v(x) dx, \qquad (1.1.10) \\ D(\mathcal{E}) &= H_0^{1,2}(D). \end{aligned}$$

By setting $a = I,\ b = 0$ and $c = 0$ off D, the operator L can be assumed to define on $\mathbf{R}^n$.

Let $X = ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbf{R}^n})$ be the Markov process associated with the following Dirichlet form

$$\begin{aligned} \mathcal{E}^0(u,v) &= \frac{1}{2} \int_{\mathbf{R}^n} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \qquad (1.1.11) \\ D(\mathcal{E}^0) &= H^{1,2}(\mathbf{R}^n). \end{aligned}$$

By Fukushima's decomposition (cf. Chapter 5 of Fukushima et al. (2011)), we have

$$X_t = x + M_t + N_t,$$

where $M_t = (M_t^1, \ldots, M_t^n)^*$ is a martingale additive functional of X with quadratic co-variation and $N_t = (N_t^1, \ldots, N_t^n)^*$ is a continuous additive functional of X locally of zero quadratic variation.

Denote by $\tau_D := \inf\{t > 0 : X_t \notin D\}$ the first exit time of X from D. Chen and Zhao (1995) proved that there exists a unique bounded continuous weak solution u to problem (1.1.6) and obtained the probabilistic representation of u.

Theorem 1.7 (Chen and Zhao (1995)) *Suppose that D is a bounded domain in $\mathbf{R}^n$ and $f \in C(\partial D)$. Then*

$$\begin{aligned} u(x) &= E_x\Big[\exp\Big(\int_0^{\tau_D} (A^{-1}b)(X_s)dM_s - \frac{1}{2}\int_0^{\tau_D} bA^{-1}b^*(X_s)ds \\ &\quad + \int_0^{\tau_D} c(X_s)ds\Big) f(X_{\tau_D})\Big] \end{aligned} \tag{1.1.12}$$

is the unique weak solution of $Lu = 0$ which is continuous in D and

$$\lim_{x \to y} u(x) = f(y)$$

for $y \in \partial D$ which is regular for the Laplace operator $(\frac{1}{2}\Delta, D)$.

Later, Chen and Zhang (2009) generalized the results of Chen and Zhao (1995) to the Dirichlet boundary value problem (1.1.6) for the following operator:

$$L = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} - \operatorname{div}(\hat{b}\cdot) + c(x), \tag{1.1.13}$$

which added the term $\operatorname{div}(\hat{b}\cdot)$ with $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_n)^*$ being a Borel measurable field such that

$$I_D(|b|^2 + |\hat{b}|^2 + |c|) \in K_n.$$

Note that $\operatorname{div}(\hat{b}\cdot)$ in (1.1.13) is just a formal writing since the vector field $\hat{b}$ is merely measurable hence its divergence exists only in the distributional sense. Chen and Zhang (2009) first generalized the results of Chen and Zhao (1995) under the Markov assumption

$$c - \operatorname{div}\hat{b} \le 0 \quad \text{in } \mathbf{R}^n, \tag{1.1.14}$$

i.e., $\int_{\mathbf{R}^n} c(x)\phi(x)dx + \sum_{i=1}^n \int_{\mathbf{R}^n} \hat{b}_i(x)\frac{\partial \phi}{\partial x_i}dx \le 0$ for any nonnegative $\phi \in C_0^\infty(\mathbf{R}^n)$. Then they used the time-reversal of a Girsanov transform from the random time τ_D with a certain h-transform to tackle the lower-order term $\operatorname{div}\hat{b}$ and got an explicit probabilistic representation of the solution to the boundary value problem without the Markov assumption (1.1.14).

In Chen and Zhang (2009), they used essentially the following result due to Meyers (1963):

For every $x_0 \in \mathbf{R}^n$, $R > 0$ and $p > n$, there is a constant $\varepsilon \in (0,1)$, depending only on n, R and p, such that if

$$(1-\varepsilon)I_{n\times n} \le A(x) \le I_{n\times n} \quad \text{for a.e. } x \in B_R := B(x_0, R), \tag{1.1.15}$$

then

$$\frac{1}{2}\nabla(A\nabla u) = \operatorname{div} f \tag{1.1.16}$$

in B_R has a unique weak solution in $H_0^{1,p}(B_R)$ for every $f = (f_1, \ldots, f_n) \in L^p(B_R; dx)$. Moreover, there is a constant $c > 0$ independent of f such that

$$\|\nabla u\|_{L^p(B_R;dx)} \le c\|f\|_{L^p(B_R;dx)}.$$

Define

$$\begin{aligned} Z_t \quad := \quad & \exp\left(\int_0^t (A^{-1}b)(X_s)dM_s + \left(\int_0^t (A^{-1}\hat{b})(X_s)dM_s\right)\circ r_t \right. \\ & \left. -\frac{1}{2}\int_0^t (b-\hat{b})A^{-1}(b-\hat{b})^*(X_s)ds + \int_0^t c(X_s)ds\right), \end{aligned} \tag{1.1.17}$$

where r_t is the time-reversal operators defined by

$$r_t(\omega)(s) := \begin{cases} \omega(t-s) & \text{if } \ 0 \le s \le t \\ \omega(0) & \text{if } \ s \ge t. \end{cases} \tag{1.1.18}$$

Theorem 1.8 (Chen and Zhang (2009)) *Let D be a bounded Lipschitz domain contained in some ball B_R, A be an $n \times n$ symmetric positive definitive matrix satisfying the condition (1.1.15), $|b| + |\hat{b}| \in L^p(D; dx)$ for some $p > n$, and $I_{D^c} \in K_n$. Let Z be defined in (1.1.17) and assume that $E_x[Z_{\tau_D}] < \infty$ for some $x \in D$. Then for every $f \in C(\partial D)$, there exists a unique weak solution u to $Lu = 0$ in D that is continuous on $\overline{D}$ with $u = f$ on ∂D. Moreover, the solution u admits the following representation:*

$$u(x) = E_x[Z_{\tau_D} f(X_{\tau_D})] \quad \text{for} \quad x \in D. \tag{1.1.19}$$

1.1.3 Hunt's hypothesis (H)

For a time-homogeneous Markov process X, Hunt's hypothesis (H) says that "every semipolar set of X is polar". This hypothesis plays a crucial role in the potential theory of (dual) Markov processes. To illustrate its importance, let us recall some potential-theoretic principles (cf. Blumenthal and Getoor (1968, 1970)).

Suppose that E is a locally compact space with a countable base. Let (X, P_x) and $(\hat{X}, \hat{P}_x)$ be a pair of dual standard Markov processes on E as described in Chapter VI of Blumenthal and Getoor (1968). Denote by $\mathcal{B}^n$ the family of all nearly Borel measurable subsets of E. For $D \subset E$, we define the first hitting time of D by

$$\sigma_D := \inf\{t > 0 : X_t \in D\}.$$

A set $D \subset E$ is called polar (respectively, essentially polar) if there exists a set $C \in \mathcal{B}^n$ such that $D \subset C$ and $P_x(\sigma_C < \infty) = 0$ for every $x \in E$ (respectively, almost every $x \in E$ with respect to the reference measure). D is called a thin set if there exists a set $C \in \mathcal{B}^n$ such that $D \subset C$ and $P_x(\sigma_C = 0) = 0$ for every $x \in E$. D is called semipolar if $D \subset \bigcup_{n=1}^{\infty} D_n$ for some thin sets $\{D_n\}_{n=1}^{\infty}$.

Denote by E_x the expectation with respect to P_x. Let $\alpha > 0$, a finite α-excessive function f on E is called a regular potential provided that $E_x\{e^{-\alpha T_n} f(X_{T_n})\} \to E_x\{e^{-\alpha T} f(X_T)\}$ for $x \in E$ whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T. Denote by $(U^\alpha)_{\alpha>0}$ the resolvent operators for X.

- **Bounded positivity principle** (P_α^*): If ν is a finite signed measure such that $U^\alpha \nu$ is bounded, then $\nu U^\alpha \nu \geq 0$, where $\nu U^\alpha \nu := \int_E U^\alpha \nu(x) \nu(dx)$.

- **Bounded energy principle** (E_α^*): If ν is a finite measure with compact support such that $U^\alpha \nu$ is bounded, then ν does not charge semipolar sets.

- **Bounded maximum principle** (M_α^*): If ν is a finite measure with compact support K such that $U^\alpha \nu$ is bounded, then $\sup\{U^\alpha \nu(x) : x \in E\} = \sup\{U^\alpha \nu(x) : x \in K\}$.

- **Bounded regularity principle** (R^*_α): If ν is a finite measure with compact support such that $U^\alpha\nu$ is bounded, then $U^\alpha\nu$ is regular.

- **Polarity principle** (**Hunt's hypothesis (H)**): Every semipolar set is polar.

Proposition 1.9 *Assume that all 1-excessive (equivalently, all α-excessive, $\alpha > 0$) functions are lower semicontinuous. Then*

$$(P^*_\alpha) \Leftrightarrow (E^*_\alpha) \Leftrightarrow (M^*_\alpha) \Leftrightarrow (R^*_\alpha) \Leftrightarrow (\mathrm{H}).$$

Proof. $(R^*_\alpha) \Leftrightarrow$ (H) is proved in Blumenthal and Getoor (1968) and $(M^*_\alpha) \Leftrightarrow$ (H) is proved in Blumenthal and Getoor (1970). $(P^*_\alpha) \Rightarrow (M^*_\alpha)$ is proved in Rao (1977) and $(M^*_\alpha) \Rightarrow (P^*_\alpha)$ is proved in Fitzsimmons (1990). By Proposition (2.1) of Blumenthal and Getoor (1970), $(E^*_\alpha) \Rightarrow (M^*_\alpha)$. By Proposition (5.1) of Blumenthal and Getoor (1970) and the equivalence of (M^*_α) and (H), $(M^*_\alpha) \Rightarrow (E^*_\alpha)$. □

Hunt's hypothesis (H) is also equivalent to some other important properties of Markov processes. For example, Blumenthal and Getoor (Proposition (4.1) of Blumenthal and Getoor (1970)) and Glover (Theorem (2.2) of Glover (1983)) showed that (H) holds if and only if the fine and cofine topologies differ by polar sets; Fitzsimmons and Kanda (1992) showed that (H) is equivalent to the dichotomy of capacity.

In spite of its importance, (H) has been verified only in some special situations. Some forty years ago, Getoor conjectured that essentially all Lévy processes satisfy (H).

From now on till the end of this subsection we let $(\Omega, \mathcal{F}, P)$ be a probability space and $X = (X_t)_{t\geq 0}$ be an $\mathbf{R}^n$-valued Lévy process on $(\Omega, \mathcal{F}, P)$ with Lévy-Khintchine exponent ψ, i.e.,

$$E[\exp\{i\langle z, X_t\rangle\}] = \exp\{-t\psi(z)\},\ \ z \in \mathbf{R}^n, t \geq 0,$$

where E denotes the expectation with respect to P. The classical Lévy-Khintchine formula tells us that

$$\psi(z) = i\langle a, z\rangle + \frac{1}{2}\langle z, Qz\rangle + \int_{\mathbf{R}^n} \left(1 - e^{i\langle z,x\rangle} + i\langle z, x\rangle 1_{\{|x|<1\}}\right) \mu(dx),$$

where $a \in \mathbf{R}^n$, Q is a symmetric nonnegative definite $n \times n$ matrix, and μ is a measure (called the Lévy measure) on $\mathbf{R}^n\backslash\{0\}$ satisfying $\int_{\mathbf{R}^n\backslash\{0\}}(1 \wedge |x|^2)\mu(dx) < \infty$. Hereafter, we use $\mathrm{Re}(\psi)$ and $\mathrm{Im}(\psi)$ to denote the real and imaginary parts of ψ, respectively, and use (a, Q, μ) to denote ψ.

Let us recall some important results obtained so far for Getoor's conjecture. When $n = 1$, Kesten (1969) (cf. also Bretagnolle (1971)) showed that if X is not a compound Poisson process, then every $\{x\}$ is non-polar if and only if

$$\int_0^\infty \mathrm{Re}([1 + \psi(z)]^{-1})dz < \infty.$$

Port and Stone (1969) proved that for the asymmetric Cauchy process on the line every x is regular for $\{x\}$, and thus (H) holds in this case. Further, Blumenthal and Getoor (1970) showed that all stable processes with index $\alpha \in (0, 2)$ on the line satisfy (H).

Kanda (1976) and Forst (1975) proved that (H) holds if X has bounded continuous transition densities (with respect to the Lebesgue measure dx) and the Lévy-Khintchine exponent ψ satisfies $|\mathrm{Im}(\psi)| \leq M(1+\mathrm{Re}(\psi))$ for some positive constant M. Rao (1977) gave a short proof of the Kanda-Forst theorem under the weaker condition that X has resolvent densities. In particular, for $n \geq 1$, all stable processes with index $\alpha \neq 1$ satisfy (H). Kanda (1978) proved that (H) holds for stable processes on $\mathbf{R}^n$ with index $\alpha = 1$ if we assume that the linear term vanishes. Silverstein (1977) extended the Kanda-Forst condition to the non-symmetric Dirichlet forms setting, Fitzsimmons (2001) extended it to the semi-Dirichlet forms setting and Han et al. (2011) extended it to the positivity-preserving forms setting. Glover and Rao (1986) proved that α-subordinates of general Hunt processes satisfy (H).

Theorem 1.10 (Glover and Rao (1986)) *Let $(X_t)_{t\geq 0}$ be a standard process on a locally compact space with a countable base and $(T_t)_{t\geq 0}$ be an independent subordinator satisfying Hunt's hypothesis (H). Then $(X_{T_t})_{t\geq 0}$ satisfies (H).*

Rao (1988) proved that if all 1-excessive functions of X are lower semicontinuous and $|\mathrm{Im}(\psi)| \leq (1 + \mathrm{Re}(\psi))f(1 + \mathrm{Re}(\psi))$, where f is an increasing function on $[1, \infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1}d\lambda = \infty$ for every $N \geq 1$, then X satisfies (H).

Theorem 1.11 (Rao (1988)) *Let X be a Lévy process such that all 1-excessive functions are lower semicontinuous. Suppose there is an increasing function f on $[1,\infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for any $N \geq 1$ and $|1+\psi| \leq (1+\mathrm{Re}(\psi))f(1+\mathrm{Re}(\psi))$. Then (H) holds.*

In Hu and Sun (2012), they showed that if Q is non-degenerate then X satisfies (H); if Q is degenerate then, under the assumption that $\mu(\mathbf{R}^n \backslash \sqrt{Q}\mathbf{R}^n) < \infty$, X satisfies (H) if and only if the equation

$$\sqrt{Q}y = -a - \int_{\{x\in\mathbf{R}^n\backslash\sqrt{Q}\mathbf{R}^n:\,|x|<1\}} x\mu(dx)$$

has at least one solution $y \in \mathbf{R}^n$. They also showed the following proposition for subordinators.

Proposition 1.12 (Hu and Sun (2012)) *If X is a subordinator and satisfies (H), then $d = 0$.*

1.2 Main results of the thesis

The results of Chapter 2 are taken from Sun and Zhang (2015). We will present the Lévy-Khintchine type representations of Dirichlet generators and semi-Dirichlet forms.

Let U be an open set of $\mathbf{R}^n$ and m a positive Radon measure on U such that $\mathrm{supp}[m] = U$. For $\delta > 0$, we define

$$U^\delta := \{x \in U : \inf_{y\in\partial U} |x-y| > \delta\}.$$

For the Lévy-Khintchine type representation of Dirichlet generators on U, we obtain the following theorem:

Theorem 1.13 *Suppose that $(A, D(A))$ is a generator on $L^2(U;m)$ such that A is a Dirichlet operator and $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$. Let $\delta > 0$ be a constant such that $U^\delta \neq \emptyset$. Then, we have the decomposition:*

$$\begin{aligned}(-Au, v) \;=\;& \frac{1}{2}\sum_{i,j=1}^n \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\nu_{ij}(dx) + \sum_{i=1}^n \int_{U^\delta} \frac{\partial u}{\partial x_i}(x)v(x)\nu_i^\delta(dx)\\ &+ \int_{U\times U\backslash d} \sum_{i=1}^n (y_i - x_i)\left(\frac{\partial u}{\partial y_i}(y)v(y) - \frac{\partial u}{\partial x_i}(x)v(x)\right) I_{\{|x-y|\leq\delta\}}(x,y)J(dxdy)\end{aligned}$$

$$+\int_{U\times U\setminus d} 2\left(u(y)-u(x)-\sum_{i=1}^{n}(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\right)v(y)J(dxdy)$$
$$+\int_U u(x)v(x)K(dx), \qquad \forall u,v\in C_0^\infty(U^\delta),$$

where J and K are the jumping and killing measures, respectively, $\{\nu_{ij}\}_{i,j=1}^n$ are signed Radon measures on U such that for any compact set $K\subset U$, $\nu_{ij}(K)=\nu_{ji}(K)$ and $\sum_{i,j=1}^n \xi_i\xi_j\nu_{ij}(K)\ge 0$ for all $(\xi_1,\dots,\xi_n)\in\mathbf{R}^n$, and $\{\nu_i^\delta\}_{i=1}^n$ are signed Radon measures on U^δ.

To get the Lévy-Khintchine type representation of semi-Dirichlet forms, we make the following assumption:

Assumption 1.14 *Let O be a relatively compact open set of U. Suppose that $\{f_n\}_{n=1}^\infty\subset C_0^\infty(O)$ and $f\in C_0^\infty(O)$ satisfying f_n and all of its partial derivatives converge uniformly to f and its corresponding partial derivatives as $n\to\infty$. Then, $\mathcal{E}(f,g)=\lim_{n\to\infty}\mathcal{E}(f_n,g)$ and $\mathcal{E}(g,f)=\lim_{n\to\infty}\mathcal{E}(g,f_n)$ for any $g\in C_0^\infty(U)$.*

Theorem 1.15 *Suppose that $(\mathcal{E},D(\mathcal{E}))$ is a semi-Dirichlet form on $L^2(U;m)$ such that $C_0^\infty(U)\subset D(\mathcal{E})$ and Assumption 1.14 holds. Let $\delta>0$ be a constant such that $U^\delta\neq\emptyset$. Then, we have the decomposition:*

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{2}\sum_{i,j=1}^n\int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\nu_{ij}(dx)+\sum_{i=1}^n\left\langle \Psi_i^\delta,\frac{\partial u}{\partial x_i}v\right\rangle\\ &\quad+\int_{U\times U\setminus d}\sum_{i=1}^n(y_i-x_i)\left(\frac{\partial u}{\partial y_i}(y)v(y)-\frac{\partial u}{\partial x_i}(x)v(x)\right)I_{\{|x-y|\le\delta\}}(x,y)J(dxdy)\\ &\quad+\int_{U\times U\setminus d}2\left(u(y)-u(x)-\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\right)v(y)J(dxdy)\\ &\quad+\int_U u(x)v(x)K(dx), \qquad \forall u,v\in C_0^\infty(U^\delta),\end{aligned}$$

where J and K are the jumping and killing measures, respectively, $\{\nu_{ij}\}_{i,j=1}^n$ are signed Radon measures on U such that for any compact set $K\subset U$, $\nu_{ij}(K)=\nu_{ji}(K)$ and $\sum_{i,j=1}^n \xi_i\xi_j\nu_{ij}(K)\ge 0$ for all $(\xi_1,\dots,\xi_n)\in\mathbf{R}^n$, and $\{\Psi_i^\delta\}_{i=1}^n$ are generalized functions on U^δ.

The results of Chapter 3 are taken from Chen, Ma and Zhang (Chen et al. (2016)). We will give the probabilistic representations of the solution of the Dirichlet boundary value problem and the non-symmetric semigroup $\{T_t\}_{t\geq 0}$ for a general class of second order non-symmetric elliptic operators L given as follows:

$$Lu = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + (c(x) - \operatorname{div}\hat{b}(x))u.$$

Theorem 1.16 *Let $n \geq 1$, D be a bounded Lipschitz domain in $\mathbf{R}^n$ and $p > n/2$. Suppose that*

(i) $A(x) = (a_{ij}(x))_{i,j=1}^n$ satisfies

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \text{for any } \xi = (\xi_i)_{i=1}^n \in \mathbf{R}^n, x \in D$$

and

$$|a_{ij}(x)| \leq \frac{1}{\lambda} \quad \text{for any } x \in D,\ 1 \leq i,j \leq n$$

for some constant $0 < \lambda \leq 1$.

(ii) $|b|^2 \in L^{p\vee 1}(D;dx)$ and $|\hat{b}|^2 \in L^{p\vee 1}(D;dx)$.

(iii) $c \in L^{p\vee 1}(D;dx)$ and $c - \operatorname{div}\hat{b} \leq g$ for some nonnegative function $g \in L^{p\vee 1}(D;dx)$ in the distributional sense.

Then, there exists a constant $M > 0$ such that whenever $\|g\|_{L^{p\vee 1}} \leq M$, for any $f \in C(\partial D)$, there exists a unique weak solution u to $Lu = 0$ in D that is continuous on $\overline{D}$ with $u = f$ on ∂D. Moreover, the solution u admits the following representation for q.e. $x \in D$,

$$\begin{aligned} u(x) &= E_x\Bigg[\exp\Bigg(\int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^{\tau_D} b^*\tilde{a}^{-1}b(X_s)ds \\ &\qquad + \int_0^{\tau_D} c(X_s)ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D} \hat{b}^H(X_s)ds\Bigg) f(X_{\tau_D})\Bigg]. \end{aligned}$$

Theorem 1.17 *For any $f, g \in L^2(D;dx)$, we have*

$$\begin{aligned} &\int_D f(x)T_t g(x)dx \\ &= E_m\Bigg[f(X_0)g(X_t)\exp\Bigg(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \\ &\qquad + \int_0^t c(X_s)ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s)ds\Bigg); t < \tau_D\Bigg]. \end{aligned}$$

The results of Chapter 4 are taken from Hu, Sun and Zhang (Hu et al. (2015)). We first present a comparison result on Lévy processes.

Let X be a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a, Q, μ). Suppose that μ_1 is a finite measure on $\mathbf{R}^n \backslash \{0\}$ such that $\mu_1 \leq \mu$. Denote $\mu_2 := \mu - \mu_1$ and let X' be a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a', Q, μ_2), where

$$a' := a + \int_{\{|x|<1\}} x\mu_1(dx).$$

Theorem 1.18 *Let X and X' be Lévy processes defined as above. Then*

(i) they have same semipolar sets.

(ii) they have same essentially polar sets.

(iii) if both X and X' have resolvent densities, then X satisfies (H) if and only if X' satisfies (H).

We also give a new necessary and sufficient condition for (H) and obtain an extended Kanda-Forst-Rao theorem.

Theorem 1.19 *Let f be an increasing function on $[1, \infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for some $N \geq 1$. Then (H) holds if and only if*

$$\lim_{\lambda\to\infty} \sum_{k=1}^{\infty} \int_{\{B(z)>A(z)f(A(z)),\, k\leq \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, A(z)\leq\lambda<(k+1)|\mathrm{Im}\psi(z)|\}} \frac{\lambda}{\lambda^2 + (\mathrm{Im}\psi(z))^2} |\hat{\nu}(z)|^2 dz = 0$$

for any finite measure ν with compact support such that $U^1\nu$ is bounded.

Theorem 1.20 *(H) holds if the following extended Kanda-Forst-Rao condition ((EKFR) for short) holds:*

(EKFR) There are two measurable functions ψ_1 and ψ_2 on $\mathbf{R}^n$ such that $\mathrm{Im}(\psi) = \psi_1 + \psi_2$, and

$$|\psi_1| \leq Af(A),$$
$$\int_{\mathbf{R}^n} \frac{|\psi_2(z)|}{(1 + \mathrm{Re}\psi(z))^2 + (\mathrm{Im}\psi(z))^2} dz < \infty,$$

where f is an increasing function on $[1, \infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for some $N \geq 1$.

Chapter 2

Lévy-Khintchine type representations of Dirichlet generators and semi-Dirichlet forms

Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbf{R}^n$. By the celebrated Lévy-Khintchine formula, we know that the infinitesimal generator A of $(X_t)_{t\geq 0}$ is characterized by (cf. Theorem 31.5 of Sato (1999))

$$\begin{aligned} Au(y) = &\frac{1}{2}\sum_{i,j=1}^{n} Q_{ij}\frac{\partial^2 u}{\partial y_i \partial y_j}(y) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial y_i}(y) \\ &+ \int_{\mathbf{R}^n}\left(u(y+x) - u(y) - \sum_{i=1}^{n} x_i \frac{\partial u}{\partial y_i}(y) I_{\{|x|\leq 1\}}(x)\right)\nu(dx), \end{aligned} \tag{2.0.1}$$

for $u \in C_0^\infty(\mathbf{R}^n)$, where $Q = (Q_{ij})_{1\leq i,j\leq n}$ is a symmetric nonnegative-definite $n\times n$ matrix, $(b_1,\ldots,b_n) \in \mathbf{R}^n$, and ν is a Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbf{R}^n}(1\wedge |x|^2)\nu(dx) < \infty$.

The decomposition of type (2.0.1) also holds for Feller processes on $\mathbf{R}^n$. Courrège (1965/66) proved that if A is a linear operator from $C_0^\infty(\mathbf{R}^n)$ to $C(\mathbf{R}^n)$ satisfying the positive maximum principle, then A is decomposed as

$$Au(y) \;=\; \frac{1}{2}\sum_{i,j=1}^{n} q_{ij}(y)\frac{\partial^2 u}{\partial y_i \partial y_j}(y) + \sum_{i=1}^{n} l_i(y)\frac{\partial u}{\partial y_i}(y) + \gamma(y)u(y)$$

$$+\int_{\mathbf{R}^n}\left(u(y+x)-u(y)w(x)-\sum_{i=1}^n x_i\frac{\partial u}{\partial y_i}(y)w(x)\right)\mu(y,dx),$$

where $\sum_{i,j=1}^n q_{ij}(y)\xi_i\xi_j \geq 0$ for all $y \in \mathbf{R}^n$ and $(\xi_1,\ldots,\xi_n) \in \mathbf{R}^n$, the function $y \to \sum_{i,j=1}^n q_{ij}(y)\xi_i\xi_j$ is upper semicontinuous, $l_i \in C(\mathbf{R}^n)$ for $1 \leq i \leq n$, $\gamma \in C(\mathbf{R}^n)$ with $\gamma \leq 0$, μ is a kernel on $\mathbf{R}^n \times \mathcal{B}(\mathbf{R}^n)$, and $w \in C_0^\infty(\mathbf{R}^n)$ with $0 \leq w \leq 1$ and $w = 1$ on $\{x \in \mathbf{R}^n : |x| \leq 1\}$ (cf. Jacob (2001) §4.5).

Suppose now that $(X_t)_{t\geq 0}$ is a general right continuous Markov process on $\mathbf{R}^n$ or, more generally, on an open set U of $\mathbf{R}^n$. Denote by $(P_t)_{t>0}$ the transition semigroup of $(X_t)_{t\geq 0}$. Suppose that there is a positive Radon measure m on U such that $(P_t)_{t>0}$ acts as a strongly continuous contraction semigroup on $L^2(U;m)$. Note that this condition is fulfilled if, for example, m is an excessive measure of $(X_t)_{t\geq 0}$. Denote by $(A, D(A))$ the L^2-generator of $(P_t)_{t>0}$. Then, $(A, D(A))$ is a Dirichlet operator, i.e., $(Au,(u-1)\vee 0) \leq 0$ for all $u \in D(A)$ (cf. Proposition I.4.3 of Ma and Röckner (1992))

Denote by $(\hat{A}, D(\hat{A}))$ the co-generator of the semigroup $(P_t)_{t>0}$. Note that generally $(\hat{A}, D(\hat{A}))$ may not be a Dirichlet operator (see Remark 2.2(ii) of Ma et al. (1995) for an example). We assume that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ and consider the following bilinear form

$$\mathcal{E}(u,v) := (-Au, v) \quad \text{for } u,v \in C_0^\infty(U). \tag{2.0.2}$$

Denote by $(G_\beta)_{\beta>0}$ and $(\hat{G}_\beta)_{\beta>0}$ the resolvent and co-resolvent of $(P_t)_{t>0}$, respectively. Similar to Hu et al. (2006) §2 (cf. also Fukushima et al. (2011) §3.2) and noting that the sector condition is not used therein, we can prove the following lemma by virtue of the fact that $\mathcal{E}(u,v) = \lim_{\beta\to\infty}\beta(u-\beta G_\beta u, v)$ for $u,v \in C_0^\infty(U)$.

Lemma 2.1 *The following statements hold.*

(i) For $\beta > 0$, there exist unique positive Radon measures σ_β and $\hat{\sigma}_\beta$ on $U \times U$ satisfying

$$(\beta G_\beta u, v) = \int_{U\times U} u(x)v(y)\sigma_\beta(dxdy) \tag{2.0.3}$$

and

$$(\beta \hat{G}_\beta u, v) = \int_{U\times U} u(x)v(y)\hat{\sigma}_\beta(dxdy)$$

for $u, v \in L^2(U; m)$*.*

(ii) There exist a unique positive Radon measure J *on* $U \times U$ *off the diagonal* d *and a unique positive Radon measure* K *on* U *such that for* $v \in C_0^\infty(U)$ *and* $u \in \{g \in C_0^\infty(U) :$ g is constant on a neighbourhood of $\mathrm{supp}[v]\}$,

$$\mathcal{E}(u, v) = \int_{U \times U \setminus d} 2(u(y) - u(x))v(y)J(dxdy) + \int_U u(x)v(x)K(dx). \tag{2.0.4}$$

Hereafter $\mathrm{supp}[u]$ *denotes the support of* u*. The measures* J *and* K *are called the jumping and killing measures, respectively.*

(iii) We have

$$(\beta/2)\sigma_\beta \to J$$

and

$$(\beta/2)\hat{\sigma}_\beta \to \hat{J}$$

vaguely on $U \times U \setminus d$ *as* $\beta \to \infty$*, where* $\hat{J}(dxdy) := J(dydx)$*.*

2.1 Lévy-Khintchine type representations of Dirichlet generators and semi-Dirichlet forms on open sets of $\mathbf{R}^n$

Throughout this section, we let U be an open set of $\mathbf{R}^n$ which is equipped with the subspace topology of $\mathbf{R}^n$ and let m be a positive Radon measure on U such that $\mathrm{supp}[m] = U$. We will give a Lévy-Khintchine type representation for Dirichlet generators and semi-Dirichlet forms on U.

Let J be the jumping measure given in Lemma 2.1. We choose a sequence of relatively compact open sets $\Omega_l \uparrow U$ and a sequence of numbers $\varsigma_l \downarrow 0$ such that the set $\Gamma_l := \{(x, y) \in \Omega_l \times \Omega_l : |x - y| \geq \varsigma_l\}$ is a continuous set w.r.t. J for every $l \in \mathbf{N}$. Hereafter when we say that a set B is a relatively compact set of an open set V of $\mathbf{R}^n$, we mean that $B \subset V$ and B

is relatively compact w.r.t. the subspace topology of V inherited from $\mathbf{R}^n$. Denote

$$\Lambda_l := \{(x, y) \in \Omega_l \times \Omega_l : |x - y| < \varsigma_l\}.$$

For $\delta > 0$, we define

$$U^\delta := \{x \in U : \inf_{y \in \partial U} |x - y| > \delta\}.$$

Hereafter, for $B \subset \mathbf{R}^n$, we denote by ∂B its boundary in $\mathbf{R}^n$.

Now we can state the first main result of this chapter.

Theorem 2.2 *Suppose that $(A, D(A))$ is a generator on $L^2(U; m)$ such that A is a Dirichlet operator and $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$. Let $\delta > 0$ be a constant such that $U^\delta \neq \emptyset$. Then, we have the decomposition:*

$$\begin{aligned}
(-Au, v) \;=\; & \frac{1}{2}\sum_{i,j=1}^{n} \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\nu_{ij}(dx) + \sum_{i=1}^{n}\int_{U^\delta}\frac{\partial u}{\partial x_i}(x)v(x)\nu_i^\delta(dx) \\
& + \int_{U\times U\backslash d}\sum_{i=1}^{n}(y_i - x_i)\left(\frac{\partial u}{\partial y_i}(y)v(y) - \frac{\partial u}{\partial x_i}(x)v(x)\right) I_{\{|x-y|\leq\delta\}}(x,y)J(dxdy) \\
& + \int_{U\times U\backslash d} 2\left(u(y) - u(x) - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\leq\delta\}}(x,y)\right)v(y)J(dxdy) \\
& + \int_U u(x)v(x)K(dx), \qquad \forall u, v \in C_0^\infty(U^\delta), \qquad (2.1.1)
\end{aligned}$$

where J and K are the jumping and killing measures, respectively, $\{\nu_{ij}\}_{i,j=1}^n$ are signed Radon measures on U such that for any compact set $K \subset U$, one has $\nu_{ij}(K) = \nu_{ji}(K)$ and $\sum_{i,j=1}^n \xi_i\xi_j\nu_{ij}(K) \geq 0$ for all $(\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, and $\{\nu_i^\delta\}_{i=1}^n$ are signed Radon measures on U^δ.

The representation (2.1.1) improves our understanding of Markov processes and has many potential applications. For example, it sheds light on the long-standing open problem, "when does a Markov process satisfy Hunt's hypothesis (H)?". For a dual diffusion on an open set of $\mathbf{R}^n$, (2.1.1) indicates the strong connection between Hunt's hypothesis (H) and the condition that the diffusion is locally associated with a semi-Dirichlet form. Here we would like to point out that Theorem 2.2 does not assume the sector condition although its proof is motivated by the theory of Dirichlet forms, and that the assumption $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ is reasonable for many applications, for example, when the martingale problem of Markov processes is studied (cf. Chapter 4 of Ethier and Kurtz (1986)).

If the diffusion part of $(X_t)_{t\geq 0}$ corresponds to a differential operator with very singular coefficients, then it is not suitable to assume that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ anymore. In this case, we will adopt the framework of semi-Dirichlet forms to investigate the analytic structure of $(X_t)_{t\geq 0}$. Suppose that $(A, D(A))$ satisfies the sector condition, i.e., there exists a positive constant κ such that

$$|((1-A)u, v)| \leq \kappa((1-A)u, u)^{1/2}((1-A)v, v)^{1/2}, \quad \forall u, v \in D(A). \tag{2.1.2}$$

Note that $(A, D(A))$ satisfies the sector condition (2.1.2) if and only if $(P_t)_{t>0}$ is an analytic semigroup (cf. Corollary I.2.21 of Ma and Röckner (1992)). Denote by $(\mathcal{E}, D(\mathcal{E}))$ the semi-Dirichlet form obtained by completing $D(A)$ w.r.t. the $((1-A)u, u)^{1/2}$-norm. Assume that $C_0^\infty(U) \subset D(\mathcal{E})$. Then, one finds that Lemma 2.1 also holds for $(\mathcal{E}, D(\mathcal{E}))$. We make the following assumption.

Assumption 2.3 *Let O be a relatively compact open set of U. Suppose that $\{f_n\}_{n=1}^\infty \subset C_0^\infty(O)$ and $f \in C_0^\infty(O)$ satisfying f_n and all of its partial derivatives converge uniformly to f and its corresponding partial derivatives as $n \to \infty$. Then, one has $\mathcal{E}(f, g) = \lim_{n\to\infty} \mathcal{E}(f_n, g)$ and $\mathcal{E}(g, f) = \lim_{n\to\infty} \mathcal{E}(g, f_n)$ for any $g \in C_0^\infty(U)$.*

We will obtain the following Lévy-Khintchine type representation of semi- Dirichlet forms, which generalizes the classical Beurling-Deny formula of symmetric Dirichlet forms on open sets of $\mathbf{R}^n$.

Theorem 2.4 *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a semi-Dirichlet form on $L^2(U; m)$ such that $C_0^\infty(U) \subset D(\mathcal{E})$ and Assumption 2.3 holds. Let $\delta > 0$ be a constant such that $U^\delta \neq \emptyset$. Then, we have the decomposition:*

$$\begin{aligned}\mathcal{E}(u, v) \;=\;& \frac{1}{2}\sum_{i,j=1}^n \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\nu_{ij}(dx) + \sum_{i=1}^n \left\langle \Psi_i^\delta, \frac{\partial u}{\partial x_i} v \right\rangle \\ &+ \int_{U\times U\backslash d} \sum_{i=1}^n (y_i - x_i)\left(\frac{\partial u}{\partial y_i}(y)v(y) - \frac{\partial u}{\partial x_i}(x)v(x)\right) I_{\{|x-y|\leq\delta\}}(x, y)J(dxdy)\end{aligned}$$

$$
\begin{aligned}
&+\int_{U\times U\backslash d} 2\left(u(y)-u(x)-\sum_{i=1}^{n}(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\right)v(y)J(dxdy)\\
&+\int_U u(x)v(x)K(dx), \qquad \forall u,v\in C_0^\infty(U^\delta), \qquad (2.1.3)
\end{aligned}
$$

where J and K are the jumping and killing measures, respectively, $\{\nu_{ij}\}_{i,j=1}^n$ are signed Radon measures on U such that for any compact set $K\subset U$, $\nu_{ij}(K)=\nu_{ji}(K)$ and $\sum_{i,j=1}^n \xi_i\xi_j\nu_{ij}(K)\ge 0$ for all $(\xi_1,\ldots,\xi_n)\in\mathbf{R}^n$, and $\{\Psi_i^\delta\}_{i=1}^n$ are generalized functions on U^δ.

We will prove Theorems 2.2 and 2.4 in Subsection 2.1.3. If Assumption 2.3 is replaced by the assumption that $(\mathcal{E},D(\mathcal{E}))$ is locally controlled by Dirichlet forms, then we can obtain a clearer characterization of the generalized functions $\{\Psi_i^\delta\}_{i=1}^n$ given in Theorem 2.4, see Corollary 2.13 below.

2.1.1 Decomposition of $\mathcal{E}$

Lemma 2.5 *Let $u,v\in C_0^\infty(U)$ and F be a compact set of U. Then*

(i) One has

$$\int_{U\times F\backslash d}(u(y)-u(x))^2J(dxdy)<\infty.$$

(ii) One has

$$\int_{F\times F\backslash d}|x-y|^2J(dxdy)<\infty.$$

(iii) For $\varepsilon>0$,

$$\int_{(U\times U)\cap\{|x-y|>\varepsilon\}}|(u(y)-u(x))v(y)|J(dxdy)<\infty.$$

Proof. (i) We choose a $w\in C_0^\infty(U)$ satisfying $w\ge 0$ and $w|_F\equiv 1$. By (2.0.3) and the sub-Markovian property of $(G_\beta)_{\beta>0}$, we get

$$
\begin{aligned}
\int_{U\times F\backslash d}(u(y)-u(x))^2J(dxdy) &\le \int_{U\times U\backslash d}(u(y)-u(x))^2w(y)J(dxdy)\\
&= \lim_{l\to\infty}\int_{\Gamma_l}(u(y)-u(x))^2w(y)J(dxdy)\\
&= \lim_{l\to\infty}\lim_{\beta\to\infty}\frac{\beta}{2}\int_{\Gamma_l}(u(y)-u(x))^2w(y)\sigma_\beta(dxdy)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lim_{\beta\to\infty} \frac{\beta}{2} \int_{U\times U} (u(y)-u(x))^2 w(y)\sigma_\beta(dxdy) \\
&= \lim_{\beta\to\infty} \frac{\beta}{2} \{(\beta G_\beta I_U, u^2 w) - 2(\beta G_\beta u, uw) + (\beta G_\beta u^2, w)\} \\
&\leq \lim_{\beta\to\infty} \left\{ \beta(u-\beta G_\beta u, uw) - \frac{\beta}{2}(u^2 - \beta G_\beta u^2, w) \right\} \\
&= \mathcal{E}(u, uw) - \frac{1}{2}\mathcal{E}(u^2, w) \\
&< \infty.
\end{aligned}
$$

(ii) We choose a $w' \in C_0^\infty(U)$ satisfying $w'|_F \equiv 1$. For $1 \leq i \leq n$, we define $u_i(x) = x_i \cdot w'(x)$ for $x = (x_1, \ldots, x_n) \in U$. Then, $u_i \in C_0^\infty(U)$ satisfying $u_i(x) = x_i$ for $x \in F$. By (i), we get

$$
\begin{aligned}
\int_{F\times F\setminus d} |x-y|^2 J(dxdy) &= \sum_{i=1}^n \int_{F\times F\setminus d} (x_i - y_i)^2 J(dxdy) \\
&= \sum_{i=1}^n \int_{F\times F\setminus d} (u_i(x) - u_i(y))^2 J(dxdy) \\
&< \infty.
\end{aligned}
$$

(iii) By (i), we get

$$
\begin{aligned}
&\int_{(U\times U)\cap\{|x-y|>\varepsilon\}} |(u(y)-u(x))v(y)| J(dxdy) \\
&= \int_{(U\times \mathrm{supp}[v])\cap\{|x-y|>\varepsilon\}} |(u(y)-u(x))v(y)| J(dxdy) \\
&= \int_{(U\times \mathrm{supp}[v])\cap\{|x-y|>\varepsilon\}} |(u(y)-u(x))(v(y)-v(x)) + (u(y)-u(x))v(x)| J(dxdy) \\
&\leq \int_{U\times \mathrm{supp}[v]\setminus d} |(u(y)-u(x))(v(y)-v(x))| J(dxdy) \\
&\quad + \int_{(\mathrm{supp}[v]\times \mathrm{supp}[v])\cap\{|x-y|>\varepsilon\}} |(u(y)-u(x))v(x)| J(dxdy) \\
&\leq \left(\int_{U\times \mathrm{supp}[v]\setminus d} (u(y)-u(x))^2 J(dxdy) \right)^{1/2} \left(\int_{U\times \mathrm{supp}[v]\setminus d} (v(y)-v(x))^2 J(dxdy) \right)^{1/2} \\
&\quad + 2\|u\|_\infty \|v\|_\infty J((\mathrm{supp}[v]\times \mathrm{supp}[v]) \cap \{|x-y|>\varepsilon\}) \\
&< \infty.
\end{aligned}
$$

$\square$

Let $\delta > 0$ be a constant such that $U^\delta \neq \emptyset$. Suppose that $u, v \in C_0^\infty(U^\delta)$. Let $\chi \in C_0^\infty(U)$ satisfying $\chi = 1$ on a neighbourhood of $\mathrm{supp}[u] \cup \mathrm{supp}[v]$. By Taylor's theorem and Lemma 2.5(ii), one finds that

$$(u(y) - u(x) - \sum_{i=1}^n (y_i - x_i) \frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq\delta\}}(x,y))v(y)\chi(x)$$

is integrable w.r.t. both J and $\hat{J}$. Hereafter, we define

$$F_v^\delta := \left\{ x \in U : \inf_{y \in \mathrm{supp}[v]} |x-y| \leq \delta \right\}. \tag{2.1.4}$$

Observe that F_v^δ is a compact set of U. By Lemma 2.5(i) and (ii), for $1 \leq i \leq n$, we have

$$\begin{aligned}
&\int_{U\times U\setminus d} \left| (y_i - x_i)\frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq\delta\}}(x,y)v(y)(1-\chi(x)) \right| (J(dxdy) + \hat{J}(dxdy)) \\
\leq\ & 2 \left\| \frac{\partial u}{\partial y_i} \cdot v \right\|_\infty \left(\int_{F_v^\delta \times F_v^\delta \setminus d} |x-y|^2 J(dxdy) \right)^{1/2} \left(\int_{F_v^\delta\times F_v^\delta\setminus d} (\chi(y) - \chi(x))^2 J(dxdy) \right)^{1/2} \\
<\ & \infty.
\end{aligned}$$

Hence

$$\sum_{i=1}^n (y_i - x_i)\frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq\delta\}}(x,y)v(y)(1-\chi(x))$$

is integrable w.r.t. both J and $\hat{J}$. Therefore,

$$\left((u(y) - u(x))\chi(x) - \sum_{i=1}^n (y_i - x_i)\frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\leq\delta\}}(x,y) \right) v(y)$$

is integrable w.r.t. both J and $\hat{J}$.

We assume temporarily that $J(\{(x,y) \in U \times U : |x-y| = \delta\}) = 0$. Then, we obtain by the vague convergence of $(\beta/2)\sigma_\beta$ to J that

$$\begin{aligned}
&\mathcal{E}(u,v) = \lim_{\beta\to\infty} \beta(u - \beta G_\beta u, v) \\
=\ & \lim_{\beta\to\infty} \beta \left\{ \int_{U\times U} (u(y) - u(x))v(y)\chi(x)\sigma_\beta(dxdy) \right. \\
&\quad \left. + \int_U \chi(x)u(x)v(x)m(dx) - \int_{U\times U} \chi(x)u(y)v(y)\sigma_\beta(dxdy) \right\} \\
=\ & \lim_{\beta\to\infty} \beta \int_{U\times U} (u(y) - u(x))v(y)\chi(x)\sigma_\beta(dxdy) + \mathcal{E}(\chi, uv)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\left\{\int_{\Lambda_l}(u(y)-u(x))v(y)\sigma_\beta(dxdy)\right.\\
&\qquad+\int_{\Gamma_l}\left((u(y)-u(x))\chi(x)-\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\right)v(y)\sigma_\beta(dxdy)\\
&\qquad\left.+\int_{\Gamma_l}\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)v(y)\sigma_\beta(dxdy)\right\}+\mathcal{E}(\chi,uv)\\
&= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\left\{\int_{\Lambda_l}(u(y)-u(x))v(y)\sigma_\beta(dxdy)\right.\\
&\qquad\qquad\left.+\int_{\Gamma_l}\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)v(y)\sigma_\beta(dxdy)\right\}\\
&\quad+\int_{U\times U\backslash d}2\left((u(y)-u(x))\chi(x)-\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\right)v(y)J(dxdy)\\
&\quad+\mathcal{E}(\chi,uv).
\end{aligned}
\tag{2.1.5}
$$

Similarly, we get

$$
\begin{aligned}
\hat{\mathcal{E}}(u,v) &= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\left\{\int_{\Lambda_l}(u(y)-u(x))v(y)\hat\sigma_\beta(dxdy)\right.\\
&\qquad\left.+\int_{\Gamma_l}\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)v(y)\hat\sigma_\beta(dxdy)\right\}\\
&\quad+\int_{U\times U\backslash d}2\left((u(y)-u(x))\chi(x)\right.\\
&\quad\left.-\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\right)v(y)\hat{J}(dxdy)+\hat{\mathcal{E}}(\chi,uv).
\end{aligned}
\tag{2.1.6}
$$

By (2.1.5) and (2.1.6), we can introduce the following definition.

Definition 2.6 *Let $\{\delta_n\}_{n=1}^\infty$ be a sequence of constants satisfying $\delta=\lim_{n\to\infty}\delta_n$, $\delta_n\ge\delta$ and $J(\{(x,y)\in U\times U:|x-y|=\delta_n\})=0$ for each $n\in\mathbf{N}$. For $u,v\in C_0^\infty(U^\delta)$, we define*

$$
\begin{aligned}
\mathcal{E}^{c,\delta}(u,v) &:= \lim_{n\to\infty}\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\left\{\int_{\Lambda_l}(u(y)-u(x))v(y)\sigma_\beta(dxdy)\right.\\
&\qquad\left.+\int_{\Gamma_l}\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta_n\}}(x,y)v(y)\sigma_\beta(dxdy)\right\}
\end{aligned}
\tag{2.1.7}
$$

and

$$
\hat{\mathcal{E}}^{c,\delta}(u,v) := \lim_{n\to\infty}\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\left\{\int_{\Lambda_l}(u(y)-u(x))v(y)\hat\sigma_\beta(dxdy)\right.
$$

$$+\int_{\Gamma_l}\sum_{i=1}^{n}(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\leq\delta_n\}}(x,y)v(y)\hat{\sigma}_\beta(dxdy)\Bigg\}. \tag{2.1.8}$$

By (2.1.5), (2.1.6) and the fact that J is a positive Radon measure J on $U\times U\backslash d$, one finds that the definitions of $\mathcal{E}^{c,\delta}$ and $\hat{\mathcal{E}}^{c,\delta}$ are independent of the selections of $\{\Omega_l\}$ and $\{\delta_n\}$. Both $\mathcal{E}^{c,\delta}(u,v)$ and $\hat{\mathcal{E}}^{c,\delta}(u,v)$ satisfy the left strong local property in the sense that $\mathcal{E}^{c,\delta}(u,v)=\hat{\mathcal{E}}^{c,\delta}(u,v)=0$ whenever u is constant on a neighbourhood of $\mathrm{supp}[v]$.

Theorem 2.7 *Suppose $u,v\in C_0^\infty(U^\delta)$.*

(i) We have the decomposition

$$\begin{aligned}\mathcal{E}(u,v)&=\mathcal{E}^{c,\delta}(u,v)\\&+\int_{U\times U\backslash d}2\left(u(y)-u(x)-\sum_{i=1}^{n}(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\leq\delta\}}(x,y)\right)v(y)J(dxdy)\\&+\int_U u(x)v(x)K(dx).\end{aligned} \tag{2.1.9}$$

(ii) Let $\chi\in C_0^\infty(U)$ satisfying $\chi=1$ on a neighbourhood of $\mathrm{supp}[u]\cup\mathrm{supp}[v]$. *Then, we have*

$$\begin{aligned}\mathcal{E}(u,v)&=\mathcal{E}^{c,\delta}(u,v)\\&+\int_{U\times U\backslash d}2\left((u(y)-u(x))\chi(x)-\sum_{i=1}^{n}(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\leq\delta\}}(x,y)\right)v(y)J(dxdy)\\&+\mathcal{E}(\chi,uv)\end{aligned} \tag{2.1.10}$$

and

$$\begin{aligned}\hat{\mathcal{E}}(u,v)&=\hat{\mathcal{E}}^{c,\delta}(u,v)\\&+\int_{U\times U\backslash d}2\left((u(y)-u(x))\chi(x)-\sum_{i=1}^{n}(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\leq\delta\}}(x,y)\right)v(y)\hat{J}(dxdy)\\&+\hat{\mathcal{E}}(\chi,uv).\end{aligned} \tag{2.1.11}$$

Proof. Statement (ii) is a direct consequence of (2.1.5)–(2.1.8). We only prove (i). By (2.0.4), we have

$$\mathcal{E}(\chi,uv)=\int_{U\times U\backslash d}2(1-\chi(x))u(y)v(y)J(dxdy)+\int_U u(x)v(x)K(dx). \tag{2.1.12}$$

Here the integrability of $(1-\chi(x))u(y)v(y)$ w.r.t. J is also ensured by Lemma 2.5(iii). Then, we obtain (2.1.9) by (2.1.10) and (2.1.12). □

By (2.1.9), to understand the structure of $\mathcal{E}$, we may concentrate on the left strong local part $\mathcal{E}^{c,\delta}$.

Suppose that $u, f \in C_0^\infty(U^\delta)$. By (2.1.7), we get

$$2\mathcal{E}^{c,\delta}(u, uf) - \mathcal{E}^{c,\delta}(u^2, f) = \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(u(y)-u(x))^2 f(y)\sigma_\beta(dxdy). \tag{2.1.13}$$

Since δ is arbitrary, $\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(\varphi(y)-\varphi(x))^2 g(y)\sigma_\beta(dxdy)$ exists for any $\varphi, g \in C_0^\infty(U)$.

Let $\varphi \in C_0^\infty(U)$. For $r \in \mathbf{N}$, we choose a $w \in C_0^\infty(U)$ satisfying $w \geq 0$ and $w|_{\Omega_r} \equiv 1$. For $g \in C_0^\infty(\Omega_r)$, we obtain by the sub-Markovian property of $(G_\beta)_{\beta>0}$ that

$$\begin{aligned}
&\left|\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(\varphi(y)-\varphi(x))^2 g(y)\sigma_\beta(dxdy)\right|\\
\leq\ & \|g\|_\infty \lim_{\beta\to\infty}\beta\int_{U\times U}(\varphi(y)-\varphi(x))^2 w(y)\sigma_\beta(dxdy)\\
\leq\ & \|g\|_\infty \lim_{\beta\to\infty}\{2\beta(\varphi-\beta G_\beta\varphi, \varphi w) - \beta(\varphi^2 - \beta G_\beta\varphi^2, w)\}\\
=\ & (2\mathcal{E}(\varphi, \varphi w) - \mathcal{E}(\varphi^2, w))\|g\|_\infty.
\end{aligned}$$

Then, there exists a unique Radon measure $\mu^{r,c}_{<\varphi>}$ on Ω_r such that

$$\int_{\Omega_r} g d\mu^{r,c}_{<\varphi>} = \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(\varphi(y)-\varphi(x))^2 g(y)\sigma_\beta(dxdy),\quad \forall g \in C_0^\infty(\Omega_r).$$

It is easy to see that $\{\mu^{r,c}_{<\varphi>}\}$ is a consistent sequence of Radon measures. Therefore, we can well define the measure $\mu^c_{<\varphi>}$ by $\mu^c_{<\varphi>} = \mu^{r,c}_{<\varphi>}$ on Ω_r, which satisfies

$$\int_U g d\mu^c_{<\varphi>} = \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(\varphi(y)-\varphi(x))^2 g(y)\sigma_\beta(dxdy),\quad \forall g \in C_0^\infty(U).$$

For $\varphi, \phi \in C_0^\infty(U)$, we define

$$\mu^c_{<\varphi,\phi>} := \frac{1}{2}(\mu^c_{<\varphi+\phi>} - \mu^c_{<\varphi>} - \mu^c_{<\phi>}).$$

Then, for any $g \in C_0^\infty(U)$, we have

$$\int_U g d\mu^c_{<\varphi,\phi>} = \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(\varphi(y)-\varphi(x))(\phi(y)-\phi(x))g(y)\sigma_\beta(dxdy). \tag{2.1.14}$$

Suppose now that $u, v, f \in C_0^\infty(U^\delta)$. We obtain by (2.1.13) and (2.1.14) that

$$\int_U f d\mu^c_{<u,v>} = \mathcal{E}^{c,\delta}(u, vf) + \mathcal{E}^{c,\delta}(v, uf) - \mathcal{E}^{c,\delta}(uv, f). \tag{2.1.15}$$

Hence, for any $h \in C_0^\infty(U^\delta)$ satisfying $h|_{\mathrm{supp}[u]\cup\mathrm{supp}[v]} \equiv 1$, we have

$$\mathcal{E}^{c,\delta}(u,v) + \mathcal{E}^{c,\delta}(v,u) = \int_U h d\mu^c_{<u,v>} + \mathcal{E}^{c,\delta}(uv,h). \tag{2.1.16}$$

For $u, v \in C_0^\infty(U^\delta)$, we define a linear functional $L^\delta(u,v)$ on $C_0^\infty(U^\delta)$ by

$$< L^\delta(u,v), f >:= \frac{1}{2}(\mathcal{E}^{c,\delta}(u,vf) - \hat{\mathcal{E}}^{c,\delta}(u,vf)), \quad f \in C_0^\infty(U^\delta). \tag{2.1.17}$$

Then, for any $h \in C_0^\infty(U^\delta)$ satisfying $h|_{\mathrm{supp}[v]} \equiv 1$, we have

$$\mathcal{E}^{c,\delta}(u,v) - \hat{\mathcal{E}}^{c,\delta}(u,v) = 2 < L^\delta(u,v), h > . \tag{2.1.18}$$

Let $\chi \in C_0^\infty(U^\delta)$ satisfying $\chi = 1$ on a neighbourhood of $\mathrm{supp}[u] \cup \mathrm{supp}[v]$. Then, we obtain by (2.1.9), (2.3.1) and (2.1.18) that

$$\begin{aligned}
& \mathcal{E}^{c,\delta}(u,v) - \mathcal{E}^{c,\delta}(v,u) \\
= \;& \mathcal{E}^{c,\delta}(u,v) - \mathcal{E}(v,u) + \int_U u(x)v(x)K(dx) \\
& + \int_{U\times U\backslash d} 2\left(v(y) - v(x) - \sum_{i=1}^n (y_i - x_i)\frac{\partial v}{\partial y_i}(y) I_{\{|x-y|\le\delta\}}(x,y)\right) u(y)J(dxdy) \\
= \;& \mathcal{E}^{c,\delta}(u,v) - \hat{\mathcal{E}}^{c,\delta}(u,v) - \hat{\mathcal{E}}(\chi, uv) + \int_U u(x)v(x)K(dx) \\
& - \int_{U\times U\backslash d} 2\left((u(y) - u(x))\chi(x) - \sum_{i=1}^n (y_i - x_i)\frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\le\delta\}}(x,y)\right) v(y)\hat{J}(dxdy) \\
& + \int_{U\times U\backslash d} 2\left(v(y) - v(x) - \sum_{i=1}^n (y_i - x_i)\frac{\partial v}{\partial y_i}(y) I_{\{|x-y|\le\delta\}}(x,y)\right) u(y)J(dxdy) \\
= \;& 2 < L^\delta(u,v), \chi > - \mathcal{E}(uv,\chi) + \int_U u(x)v(x)K(dx) \\
& - \int_{U\times U\backslash d} 2\left((u(y) - u(x))\chi(x) - \sum_{i=1}^n (y_i - x_i)\frac{\partial u}{\partial y_i}(y) I_{\{|x-y|\le\delta\}}(x,y)\right) v(y)\hat{J}(dxdy) \\
& + \int_{U\times U\backslash d} 2\left(v(y) - v(x) - \sum_{i=1}^n (y_i - x_i)\frac{\partial v}{\partial y_i}(y) I_{\{|x-y|\le\delta\}}(x,y)\right) u(y)J(dxdy) \\
= \;& 2 < L^\delta(u,v), \chi > - \mathcal{E}^{c,\delta}(uv,\chi) \\
& - \int_{U\times U\backslash d} 2\left((uv)(y) - (uv)(x) - \sum_{i=1}^n (y_i - x_i)\frac{\partial (uv)}{\partial y_i}(y) I_{\{|x-y|\le\delta\}}(x,y)\right) \chi(y)J(dxdy) \\
& - \int_{U\times U\backslash d} 2\left((u(x) - u(y))\chi(y) - \sum_{i=1}^n (x_i - y_i)\frac{\partial u}{\partial x_i}(x) I_{\{|x-y|\le\delta\}}(x,y)\right) v(x)J(dxdy)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{U\times U\backslash d} 2\left(v(y)-v(x)-\sum_{i=1}^{n}(y_i-x_i)\frac{\partial v}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\right)u(y)J(dxdy)\\
=\ & 2< L^{\delta}(u,v),\chi> -\mathcal{E}^{c,\delta}(uv,\chi)\\
&+\int_{U\times U\backslash d} 2\sum_{i=1}^{n}(y_i-x_i)\left(\frac{\partial u}{\partial y_i}(y)v(y)-\frac{\partial u}{\partial x_i}(x)v(x)\right)I_{\{|x-y|\le\delta\}}(x,y)J(dxdy). \quad (2.1.19)
\end{aligned}
$$

By (2.1.16) and (2.1.19), we obtain the following theorem.

Theorem 2.8 *Suppose $u,v\in C_0^\infty(U^\delta)$ and $\chi\in C_0^\infty(U^\delta)$ satisfying $\chi=1$ on a neighbourhood of* $\mathrm{supp}[u]\cup\mathrm{supp}[v]$. *Then*

$$
\begin{aligned}
\mathcal{E}^{c,\delta}(u,v) &= \frac{1}{2}\int_U \chi d\mu^c_{<u,v>} + < L^{\delta}(u,v),\chi>\\
&+\int_{U\times U\backslash d}\sum_{i=1}^{n}(y_i-x_i)\left(\frac{\partial u}{\partial y_i}(y)v(y)-\frac{\partial u}{\partial x_i}(x)v(x)\right)I_{\{|x-y|\le\delta\}}(x,y)J(dxdy). \quad (2.1.20)
\end{aligned}
$$

2.1.2 Transformation rules for the symmetric and co-symmetric diffusion parts

In this subsection, we will derive transformation rules for the sign Radon measure $\mu^c_{<\cdot,\cdot>}$ and the lineal functional $L^\delta(\cdot,\cdot)$ introduced in Subsection 2.1.1.

Theorem 2.9 *The following statements hold.*

(i) For $u,v,w\in C_0^\infty(U)$,

$$
d\mu^c_{<uv,w>} = ud\mu^c_{<v,w>} + vd\mu^c_{<u,w>}.
$$

(ii) For $u,v,w,f\in C_0^\infty(U^\delta)$,

$$
< L^{\delta}(u,vw),f> = < L^{\delta}(u,v),wf> .
$$

(iii) For $u,v,w,f\in C_0^\infty(U^\delta)$,

$$
< L^{\delta}(uv,w),f> = < L^{\delta}(u,w),vf> + < L^{\delta}(v,w),uf> .
$$

Proof. We assume without loss of generality that $u,v,w,f\in C_0^\infty(U^\delta)$.

(i) We need only show that

$$d\mu^c_{<u^2,v>} = 2ud\mu^c_{<u,v>}.$$

To this end, we choose a $\chi \in C_0^\infty(U)$ satisfying $\chi = 1$ on a neighbourhood of $\mathrm{supp}[u] \cup \mathrm{supp}[v]$. Let $g \in C_0^\infty(U^\delta)$. Then, by (2.1.14) and Lemma 2.5(i), we get

$$\begin{aligned}
& \int_{U^\delta} g d\mu^c_{<u^2,v>} - 2\int_{U^\delta} gud\mu^c_{<u,v>} \\
= & -\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(u(y)-u(x))^2(v(y)-v(x))g(y)\sigma_\beta(dxdy) \\
= & -\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(u(y)-u(x))^2(v(y)-v(x))g(y)\chi(x)\sigma_\beta(dxdy) \\
& -\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}u^2(y)v(y)g(y)(1-\chi(x))\sigma_\beta(dxdy) \\
= & -\lim_{l\to\infty}\int_{\Lambda_l}2(u(y)-u(x))^2(v(y)-v(x))g(y)\chi(x)J(dxdy) \\
= & \ 0.
\end{aligned}$$

Statement (ii) is obvious by (2.1.17).

(iii) We need only show that

$$< L^\delta(u^2,v), f >= 2 < L^\delta(u,v), uf > .$$

By (2.1.7), (2.1.8) and (2.1.17), we get

$$\begin{aligned}
& < L^\delta(u,v), f > \\
= & \lim_{n\to\infty}\lim_{l\to\infty}\lim_{\beta\to\infty}\frac{\beta}{2}\Bigg\{\int_{\Lambda_l}(u(y)-u(x))(v(y)f(y)+v(x)f(x))\sigma_\beta(dxdy) \\
& +\int_{\Gamma_l}\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta_n\}}(x,y)v(y)f(y)\sigma_\beta(dxdy) \\
& -\int_{\Gamma_l}\sum_{i=1}^n(y_i-x_i)\frac{\partial u}{\partial y_i}(y)I_{\{|x-y|\le\delta_n\}}(x,y)v(y)f(y)\hat{\sigma}_\beta(dxdy)\Bigg\}.
\end{aligned}$$

We choose a function $\chi \in C_0^\infty(U)$ with the property that $\chi = 1$ on a neighbourhood of $\mathrm{supp}[u] \cup \mathrm{supp}[v]$. Then, by Lemma 2.5(i), we get

$$< L^\delta(u^2,v), f > -2 < L^\delta(u,v), uf >$$

$$
\begin{aligned}
&= -\lim_{l\to\infty}\lim_{\beta\to\infty}\frac{\beta}{2}\int_{\Lambda_l}(u(y)-u(x))^2(v(y)f(y)-v(x)f(x))\sigma_\beta(dxdy)\\
&= -\lim_{l\to\infty}\lim_{\beta\to\infty}\frac{\beta}{2}\int_{\Lambda_l}(u(y)-u(x))^2(v(y)f(y)-v(x)f(x))\chi(x)\chi(y)\sigma_\beta(dxdy)\\
&\quad -\lim_{l\to\infty}\lim_{\beta\to\infty}\frac{\beta}{2}\int_{\Lambda_l}u^2(y)v(y)f(y)(1-\chi(x))\sigma_\beta(dxdy)\\
&\quad +\lim_{l\to\infty}\lim_{\beta\to\infty}\frac{\beta}{2}\int_{\Lambda_l}u^2(x)v(x)f(x)(1-\chi(y))\sigma_\beta(dxdy)\\
&= -\lim_{l\to\infty}\int_{\Lambda_l}(u(y)-u(x))^2(v(y)f(y)-v(x)f(x))\chi(x)\chi(y)J(dxdy)\\
&= 0.
\end{aligned}
$$

□

Let $w\in C_0^\infty(U)$ and V be a relatively compact open set of U. If $w=k$ (constant) on V, then $\mu^c_{<w>}=0$ on V. In fact, taking an $f\in C_0^\infty(V)$, we obtain by Theorem 2.9(i) that

$$kd\mu^c_{<f,w>}=d\mu^c_{<fw,w>}=fd\mu^c_{<w>}+wd\mu^c_{<f,w>},$$

which implies that $fd\mu^c_{<w>}=0$ on V. Since $f\in C_0^\infty(V)$ is arbitrary, $\mu^c_{<w>}=0$ on V. For $u,v\in C^\infty(U)$, we choose a sequence of functions $\{u_l,v_l\}\subset C_0^\infty(U)$ such that $u=u_l$ and $v=v_l$ on Ω_l. Therefore, we can well define the measure $\mu^c_{<u,v>}$ by

$$\mu^c_{<u,v>}=\mu^c_{<u_l,v_l>}$$

on Ω_l. The definition of $\mu^c_{<u,v>}$ is independent of the selections of $\{\Omega_l\}$ and $\{u_l,v_l\}$.

For $u,v\in C^\infty(U^\delta)$, we choose a sequence of relatively compact open sets $V_l\uparrow U^\delta$ and a sequence of functions $\{u_l,v_l\}\subset C_0^\infty(U^\delta)$ such that $u=u_l$ and $v=v_l$ on V_l. By (2.1.17) and the left strong local property of $\mathcal{E}^{c,\delta}$ and $\hat{\mathcal{E}}^{c,\delta}$, we can well-define the linear functional $L^\delta(u,v)$ by

$$<L^\delta(u,v),f>=\lim_{l\to\infty}<L^\delta(u_l,v_l),f>$$

for $f\in C_0^\infty(U^\delta)$. The definition of $L^\delta(u,v)$ is independent of the selections of $\{V_l\}$ and $\{u_l,v_l\}$.

Theorem 2.10 *Let $\Phi\in C^\infty(\mathbf{R}^m)$.*

(i) For $u_1,\ldots,u_m,v,w\in C^\infty(U)$,

$$d\mu^c_{<\Phi(u_1,\ldots,u_m),v>}=\sum_{i=1}^m\Phi_{x_i}(u_1,\ldots,u_m)d\mu^c_{<u_i,v>}.$$

(ii) For $u_1,\ldots,u_m,v,w\in C^\infty(U^\delta)$ and $f\in C_0^\infty(U^\delta)$,

$$<L^\delta(\Phi(u_1,\ldots,u_m),vw),f>=\sum_{i=1}^m<L^\delta(u_i,v),\Phi_{x_i}(u_1,\ldots,u_m)wf>.$$

Proof. Since the constant function belongs to $C^\infty(U)$, to prove the theorem, we may assume without loss of generality that $\Phi\in C^\infty(\mathbf{R}^m)$ with $\Phi(0)=0$ and $u_1,\ldots,u_m,v,w,f\in C_0^\infty(U^\delta)$. To simplify notation, denote $u=(u_1,\ldots,u_m)$. Let $\mathcal{C}$ be the family of all Φ satisfying (i) and (ii). By Theorem 2.9, we know that if $\Psi,\Gamma\in\mathcal{C}$, then $\Psi\Gamma\in\mathcal{C}$. Since $\mathcal{C}$ contains the coordinate functions, it contain all polynomials vanishing at the origin.

Let V be a finite cube containing the range of the function u. Then, there exists a sequence $\{\Phi^{(k)}\}$ of polynomials vanishing at the origin such that $\Phi^{(k)}$ and all of its partial derivatives converge uniformly to Φ and its corresponding partial derivatives on V (cf. Couant and Hilbert (1953) Chapter II, §4).

(i) Let $g\in C_0^\infty(U^\delta)$. We choose a function $\phi\in C_0^\infty(U)$ satisfying $\phi=1$ on $F^\delta_{u_1}\cup\cdots\cup F^\delta_{u_m}\cup F^\delta_v\cup F^\delta_g$ (see (2.1.4) for the definition of $F^\delta_\cdot$). Then, we obtain by (2.1.10), (2.1.15), the assumption that $C_0^\infty(U)\subset D(A)\cap D(\hat{A})$ or Assumption 2.3, Taylor's theorem and Lemma 2.5(ii), the finiteness of J on $(\mathrm{supp}[\phi]\times\mathrm{supp}[\phi])\cap\{|x-y|>\delta\}$, and the dominated convergence theorem that

$$\begin{aligned}
&\int_{U^\delta}gd\mu^c_{<\Phi(u),v>}\\
=\ &\mathcal{E}^{c,\delta}(\Phi(u),vg)+\mathcal{E}^{c,\delta}(v,\Phi(u)g)-\mathcal{E}^{c,\delta}(\Phi(u)v,g)\\
=\ &\mathcal{E}(\Phi(u),vg)+\mathcal{E}(v,\Phi(u)g)-\mathcal{E}(\Phi(u)v,g)-\mathcal{E}(\phi,\Phi(u)vg)\\
&\quad-\int_{U\times U\backslash d}2\Bigg(\Phi(u)(y)-\Phi(u)(x)\\
&\qquad-\sum_{i=1}^n(y_i-x_i)\frac{\partial\Phi(u)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(vg)(y)\phi(x)J(dxdy)\\
&\quad-\int_{U\times U\backslash d}2\Bigg(v(y)-v(x)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial v}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(\Phi(u)g)(y)\phi(x)J(dxdy) \\
&\quad + \int_{U\times U\backslash d} 2\Bigg((\Phi(u)v)(y) - (\Phi(u)v)(x) \\
&\quad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial(\Phi(u)v)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)g(y)\phi(x)J(dxdy) \\
= &\lim_{k\to\infty}\Bigg\{\mathcal{E}(\Phi^{(k)}(u), vg) + \mathcal{E}(v, \Phi^{(k)}(u)g) - \mathcal{E}(\Phi^{(k)}(u)v, g) - \mathcal{E}(\phi, \Phi^{(k)}(u)vg) \\
&\quad - \int_{U\times U\backslash d} 2\Bigg(\Phi^{(k)}(u)(y) - \Phi^{(k)}(u)(x) \\
&\quad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial \Phi^{(k)}(u)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(vg)(y)\phi(x)J(dxdy) \\
&\quad - \int_{U\times U\backslash d} 2\Bigg(v(y) - v(x) \\
&\quad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial v}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(\Phi^{(k)}(u)g)(y)\phi(x)J(dxdy) \\
&\quad + \int_{U\times U\backslash d} 2\Bigg((\Phi^{(k)}(u)v)(y) - (\Phi^{(k)}(u)v)(x) \\
&\quad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial(\Phi^{(k)}(u)v)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)g(y)\phi(x)J(dxdy)\Bigg\} \\
= &\lim_{k\to\infty}\{\mathcal{E}^{c,\delta}(\Phi^{(k)}(u), vg) + \mathcal{E}^{c,\delta}(v, \Phi^{(k)}(u)g) - \mathcal{E}^{c,\delta}(\Phi^{(k)}(u)v, g)\} \\
= &\lim_{k\to\infty}\int_{U^\delta} g\, d\mu^c_{<\Phi^{(k)}(u),v>} \\
= &\lim_{k\to\infty}\sum_{i=1}^{m}\int_{U^\delta} g\Phi^{(k)}_{x_i}(u)\, d\mu^c_{<u_i,v>} \\
= &\sum_{i=1}^{m}\int_{U^\delta} g\Phi_{x_i}(u)\, d\mu^c_{<u_i,v>}.
\end{aligned}
$$

(ii) We choose a function $\phi \in C_0^\infty(U)$ satisfying $\phi = 1$ on $F^\delta_{u_1} \cup \cdots \cup F^\delta_{u_m} \cup F^\delta_v \cup F^\delta_f$. By (2.1.10), (2.3.1), (2.1.17), the assumption that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ or Assumption 2.3, Taylor's theorem and Lemma 2.5(ii), the finiteness of J on $(\mathrm{supp}[\phi]\times\mathrm{supp}[\phi])\cap\{|x-y| > \delta\}$, and the dominated convergence theorem, we get

$$
< L^\delta(\Phi(u), v), f >
$$

$$
\begin{aligned}
&= \frac{1}{2}(\mathcal{E}^{c,\delta}(\Phi(u), vf) - \hat{\mathcal{E}}^{c,\delta}(\Phi(u), vf)) \\
&= \frac{1}{2}\Big[\mathcal{E}(\Phi(u), vf) - \mathcal{E}(\phi, \Phi(u)vf) - \hat{\mathcal{E}}(\Phi(u), vf) + \hat{\mathcal{E}}(\phi, \Phi(u)vf)\Big] \\
&\qquad - \int_{U\times U\setminus d}\Bigg(\Phi(u)(y) - \Phi(u)(x) \\
&\qquad\qquad \sum_{i=1}^{n}(y_i - x_i)\frac{\partial\Phi(u)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(vf)(y)\phi(x)J(dxdy) \\
&\qquad + \int_{U\times U\setminus d}\Bigg(\Phi(u)(y) - \Phi(u)(x) \\
&\qquad\qquad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial\Phi(u)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(vf)(y)\phi(x)\hat{J}(dxdy) \\
&= \lim_{k\to\infty}\Bigg\{\frac{1}{2}\Big[\mathcal{E}(\Phi^{(k)}(u), vf) - \mathcal{E}(\phi, \Phi^{(k)}(u)vf) \\
&\qquad\quad -\hat{\mathcal{E}}(\Phi^{(k)}(u), vf) + \hat{\mathcal{E}}(\phi, \Phi^{(k)}(u)vf)\Big] \\
&\qquad - \int_{U\times U\setminus d}\Bigg(\Phi^{(k)}(u)(y) - \Phi^{(k)}(u)(x) \\
&\qquad\qquad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial\Phi^{(k)}(u)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(vf)(y)\phi(x)J(dxdy) \\
&\qquad + \int_{U\times U\setminus d}\Bigg(\Phi^{(k)}(u)(y) - \Phi^{(k)}(u)(x) \\
&\qquad\qquad - \sum_{i=1}^{n}(y_i - x_i)\frac{\partial\Phi^{(k)}(u)}{\partial y_i}(y)I_{\{|x-y|\le\delta\}}(x,y)\Bigg)(vf)(y)\phi(x)\hat{J}(dxdy)\Bigg\} \\
&= \lim_{k\to\infty}\frac{1}{2}(\mathcal{E}^{c,\delta}(\Phi^{(k)}(u), vf) - \hat{\mathcal{E}}^{c,\delta}(\Phi^{(k)}(u), vf)) \\
&= \lim_{k\to\infty} < L^{\delta}(\Phi^{(k)}(u), v), f > \\
&= \lim_{k\to\infty}\sum_{i=1}^{m} < L^{\delta}(u_i, v), \Phi^{(k)}_{x_i}(u)f > \\
&= \lim_{k\to\infty}\sum_{i=1}^{m}\frac{1}{2}(\mathcal{E}^{c,\delta}(u_i, v\Phi^{(k)}_{x_i}(u)f) - \hat{\mathcal{E}}^{c,\delta}(u_i, v\Phi^{(k)}_{x_i}(u)f)) \\
&= \lim_{k\to\infty}\sum_{i=1}^{m}\Bigg\{\frac{1}{2}\Big[\mathcal{E}(u_i, v\Phi^{(k)}_{x_i}(u)f) - \mathcal{E}(\phi, u_i v\Phi^{(k)}_{x_i}(u)f) \\
&\qquad\qquad -\hat{\mathcal{E}}(u_i, v\Phi^{(k)}_{x_i}(u)f) + \hat{\mathcal{E}}(\phi, u_i v\Phi^{(k)}_{x_i}(u)f)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_{U\times U\setminus d} \left(u_i(y) - u_i(x) - \sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{\{|x-y|\le\delta\}}(x,y) \right) \\
&\qquad\qquad \times (v\Phi_{x_i}^{(k)}(u)f)(y)\phi(x)J(dxdy) \\
&\quad + \int_{U\times U\setminus d} \left(u_i(y) - u_i(x) - \sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{\{|x-y|\le\delta\}}(x,y) \right) \\
&\qquad\qquad \left. \times (v\Phi_{x_i}^{(k)}(u)f)(y)\phi(x)\hat{J}(dxdy) \right\} \\
= & \sum_{i=1}^{m} \left\{ \frac{1}{2} \Big[\mathcal{E}(u_i, v\Phi_{x_i}(u)f) - \mathcal{E}(\phi, u_i v\Phi_{x_i}(u)f) \right. \\
&\qquad - \hat{\mathcal{E}}(u_i, v\Phi_{x_i}(u)f) + \hat{\mathcal{E}}(\phi, u_i v\Phi_{x_i}(u)f) \Big] \\
&\quad - \int_{U\times U\setminus d} \left(u_i(y) - u_i(x) - \sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{\{|x-y|\le\delta\}}(x,y) \right) \\
&\qquad\qquad \times (v\Phi_{x_i}(u)f)(y)\phi(x)J(dxdy) \\
&\quad + \int_{U\times U\setminus d} \left(u_i(y) - u_i(x) - \sum_{j=1}^{n} (y_j - x_j) \frac{\partial u_i}{\partial y_j}(y) I_{\{|x-y|\le\delta\}}(x,y) \right) \\
&\qquad\qquad \left. \times (v\Phi_{x_i}(u)f)(y)\phi(x)\hat{J}(dxdy) \right\} \\
= & \sum_{i=1}^{m} \frac{1}{2} (\mathcal{E}^{c,\delta}(u_i, v\Phi_{x_i}(u)f) - \hat{\mathcal{E}}^{c,\delta}(u_i, v\Phi_{x_i}(u)f)) \\
= & \sum_{i=1}^{m} < L^{\delta}(u_i, v), \Phi_{x_i}(u)f > .
\end{aligned}
$$

Therefore, the proof is complete by noting Theorem 2.9(ii). □

2.1.3 Proofs of Theorems 2.2 and 2.4

We first characterize the first two terms of (2.1.20). Suppose that $u, v \in C_0^\infty(U^\delta)$ and $\chi \in C_0^\infty(U^\delta)$ satisfying $\chi = 1$ on a neighbourhood of $\mathrm{supp}[u] \cup \mathrm{supp}[v]$. Denote by x_i, $1 \le i \le n$, the coordinate functions of $\mathbf{R}^n$. For $1 \le i, j \le n$, we define $\nu_{ij} := \mu^c_{<x_i,x_j>}$, which is a Radon measure on U. Then, by Theorem 2.10(i), we get

$$
\int_U \chi d\mu^c_{<u,v>} = \sum_{i,j=1}^{n} \int_U \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu_{ij}(dx). \tag{2.1.21}
$$

For $1 \le i \le n$, we define the linear functional Ψ_i^δ on $C_0^\infty(U^\delta)$ by

$$< \Psi_i^\delta, f >=< L^\delta(x_i, 1), f >, \quad f \in C_0^\infty(U^\delta). \tag{2.1.22}$$

Then, by Theorem 2.10(ii) and (2.1.22), we get

$$< L^\delta(u, v), \chi >= \sum_{i=1}^n \left\langle \Psi_i^\delta, \frac{\partial u}{\partial x_i} v \right\rangle. \tag{2.1.23}$$

We now show that each Ψ_i^δ is a generalized function on U^δ. Let O be an arbitrary relatively compact open set of U^δ. Suppose that $\{f_n\}$ is a sequence of functions in $C_0^\infty(O)$ such that f_n and all of its partial derivatives converge uniformly to some $f \in C_0^\infty(O)$ and its corresponding partial derivatives as $n \to \infty$. We fix a $\xi_i \in C_0^\infty(U^\delta)$ satisfying $\xi_i = x_i$ on O and choose a $\psi \in C_0^\infty(U)$ satisfying $\psi = 1$ on $F_{\xi_i}^\delta \cup \{x \in U : \inf_{y \in O} |x - y| \le \delta\}$. For $g \in C_0^\infty(O)$, by (2.1.10), (2.3.1), (2.1.17) and (2.1.22), we get

$$\begin{aligned}
& < \Psi_i^\delta, g > \\
= \; & < L^\delta(x_i, 1), g > \\
= \; & \frac{1}{2}(\mathcal{E}^{c,\delta}(\xi_i, g) - \hat{\mathcal{E}}^{c,\delta}(\xi_i, g)) \\
= \; & \frac{1}{2}(\mathcal{E}(\xi_i, g) - \mathcal{E}(\psi, \xi_i g) - \hat{\mathcal{E}}(\xi_i, g) + \hat{\mathcal{E}}(\psi, \xi_i g)) \\
& - \int_{U \times U \setminus d} \left(\xi_i(y) - \xi_i(x) - (y_i - x_i) I_{\{|x-y| \le \delta\}}(x, y)\right) g(y)\psi(x) J(dxdy) \\
& + \int_{U \times U \setminus d} \left(\xi_i(y) - \xi_i(x) - (y_i - x_i) I_{\{|x-y| \le \delta\}}(x, y)\right) g(y)\psi(x) \hat{J}(dxdy).
\end{aligned} \tag{2.1.24}$$

Then, we obtain by formula (2.1.24), the assumption that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ or Assumption 2.3, Taylor's theorem and Lemma 2.5(ii), the finiteness of J on $(\mathrm{supp}[\psi] \times \mathrm{supp}[\psi]) \cap \{|x - y| > \delta\}$, and the dominated convergence theorem that

$$< \Psi_i^\delta, f >= \lim_{n \to \infty} < \Psi_i^\delta, f_n > .$$

Therefore, the proof of Theorem 2.4 is complete by (2.1.9), (2.1.20), (2.1.21) and (2.1.23).

To complete the proof of Theorem 2.2, we need only show that there exist signed Radon measures $\{\nu_i^\delta\}_{i=1}^n$ on U^δ such that for each $1 \le i \le n$,

$$< \Psi_i^\delta, g >= \int_{U^\delta} g(x) \nu_i^\delta(dx), \quad \forall g \in C_0^\infty(U^\delta).$$

In fact, let O be an arbitrary relatively compact open set of U^δ. Then, by (2.1.24), the assumption that $C_0^\infty(U) \subset D(A) \cap D(\hat{A})$ and Lemma 2.5 (ii) and (iii), one finds that there exists a unique signed Radon measure ν_i^O on O such that

$$< \Psi_i^\delta, g >= \int_O g(x)\nu_i^O(dx), \quad \forall g \in C_0^\infty(O).$$

Therefore, we can well-define $\nu_i^\delta = \nu_i^O$ for each O. The proof is complete. $\square$

From now on till the end of this section, we suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a semi-Dirichlet form on $L^2(U;m)$ satisfying $C_0^\infty(U) \subset D(\mathcal{E})$.

Remark 2.11 *Assumption 2.3 is implied by the following assumption.*

Assumption 2.12 *There exist a sequence of Dirichlet forms $(\mathcal{Q}^l, D(\mathcal{Q}^l))$ on the space $L^2(\Omega_l;m)$ and a sequence of positive constants C_l such that*

$$C_0^\infty(\Omega_l) \subset D(\mathcal{Q}_l)$$

and

$$\mathcal{E}_1(g,g) \leq C_l \mathcal{Q}_1^l(g,g), \quad \forall g \in C_0^\infty(\Omega_l).$$

In fact, suppose Assumption 2.12 holds and O is a relatively compact open set of U^δ. Then, there exist an open set O_0 of U^δ satisfying $\overline{O} \subset O_0$ and a regular symmetric Dirichlet form $(\mathcal{Q}, D(\mathcal{Q}))$ on $L^2(O_0;m)$ such that $C_0^\infty(O_0) \subset D(\mathcal{Q})$ and

$$\mathcal{E}_1(g,g) \leq C\mathcal{Q}_1(g,g), \quad \forall g \in C_0^\infty(O_0), \tag{2.1.25}$$

for some positive constant C. We consider the classical Beurling-Deny formula for $(\mathcal{Q}, D(\mathcal{Q}))$ (cf. Theorem 3.2.3 of Fukushima et al. (2011)):

$$\begin{aligned}\mathcal{Q}(u,v) = {} & \frac{1}{2}\sum_{i,j=1}^n \int_{O_0} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\nu_{ij}^{\mathcal{Q}}(dx) \\ & + \int_{O_0\times O_0 \backslash d} (u(x)-u(y))(v(x)-v(y))J^{\mathcal{Q}}(dxdy) + \int_{O_0} u(x)v(x)K^{\mathcal{Q}}(dx),\end{aligned} \tag{2.1.26}$$

where $u, v \in C_0^\infty(O_0)$ and we use the superscript "$\mathcal{Q}$" to emphasize that the corresponding Radon measures are for $(\mathcal{Q}, D(\mathcal{Q}))$. Note that for any compact set K and open set O_1 with $K \subset O_1 \subset O_0$ (cf. Fukushima et al. (2011) (1.2.4)),

$$\int_{K\times K\backslash d} |x-y|^2 J^{\mathcal{Q}}(dxdy) < \infty, \quad J^{\mathcal{Q}}(K, O_0 - O_1) < \infty. \tag{2.1.27}$$

Suppose $\{f_n\}_{n=1}^\infty \subset C_0^\infty(O)$ and $f \in C_0^\infty(O)$ satisfying f_n and all of its partial derivatives converge uniformly to f and its corresponding partial derivatives as $n \to \infty$. By (2.1.26), (2.1.27) and the dominated convergence theorem, we find that f_n converges to f w.r.t. the $\tilde{\mathcal{Q}}_1^{1/2}$-norm as $n \to \infty$. Therefore, we obtain by (2.1.25) that $\lim_{n\to\infty} \mathcal{E}_1(f_n - f, f_n - f) = 0$.

Corollary 2.13 *Assume the setting of Theorem 2.4 but with Assumption 2.3 replaced by Assumption 2.12. Then, we have the decomposition given in Theorem 2.4. Moreover, for any relatively compact open set O of U^δ, there exist signed Radon measures $\{\mu_i^O\}_{i=1}^n$ and $\{\mu_{ij}^O\}_{i,j=1}^n$ on O such that for $1 \le i \le n$,*

$$< \Psi_i^\delta, g >= \int_O g(x)\mu_i^O(dx) + \sum_{j=1}^n \int_O \frac{\partial g}{\partial x_j}(x)\mu_{ij}^O(dx), \quad \forall g \in C_0^\infty(O).$$

Proof. Let O be a relatively compact open set of U^δ. By Assumption 2.12, there exist an open set O_0 of U^δ satisfying $\overline{O} \subset O_0$ and a regular symmetric Dirichlet form $(\mathcal{Q}, D(\mathcal{Q}))$ on $L^2(O_0; m)$ such that $C_0^\infty(O_0) \subset D(\mathcal{Q})$ and (2.1.25) holds.

By (2.1.24), (2.1.25), the sector condition and Lemma 2.5 (ii) and (iii), to prove the corollary, we need only show that for any $u \in D(\mathcal{Q})$ there exist signed Radon measures μ^u and $\{\mu_j^u\}_{j=1}^n$ on O such that

$$\mathcal{Q}(u, v) = \int_O v(x)\mu^u(dx) + \sum_{j=1}^n \int_O \frac{\partial v}{\partial x_j}(x)\mu_j^u(dx), \quad \forall v \in C_0^\infty(O).$$

By Theorems 3.2.2 and 5.3.1 of Fukushima et al. (2011), we get

$$\begin{aligned}\mathcal{Q}(u, v) = &\frac{1}{2}\sum_{j=1}^n \int_{O_0} \frac{\partial v}{\partial x_j}\mu^c_{<u,\xi_j>}(dx)\\ &+ \int_{O_0\times O_0\backslash d} (\tilde{u}(x) - \tilde{u}(y))(v(x) - v(y))J^{\mathcal{Q}}(dxdy) + \int_{O_0} u(x)v(x)K^{\mathcal{Q}}(dx),\end{aligned}$$

where $\xi_j \in C_0^\infty(U^\delta)$ satisfying $\xi_j = x_j$ on O for $1 \leq j \leq n$ as in (2.1.24), μ^c denotes the local part of the energy measure of $(\mathcal{Q}, D(\mathcal{Q}))$, $\tilde{u}$ denotes a quasi-continuous version of u. Therefore, the proof is complete by the mean value theorem, (2.1.27) and the Riesz representation theorem. □

2.2 Examples

Our Theorems 2.2 and 2.4 show that, under suitable conditions, any Dirichlet generator and semi-Dirichlet form on $\mathbf{R}^n$ have the representations (2.1.1) and (2.1.3), respectively. On the other hand, many authors have studied Markov processes on $\mathbf{R}^n$ by using the Lévy-Khintchine type representation. Their results provide non-trivial examples for Theorems 2.2 and 2.4. In this section, we give two examples appearing in recent papers.

Example 2.14 (see Uemura (2014b), cf. also Schilling and Wang (2015)) *We consider the stable-like processes introduced by Bass* (see Bass (1988))*. Take $\alpha \in C_b^2(\mathbf{R}^n)$. Assume that there exist positive constants $\underline{\alpha}$ and $\overline{\alpha}$ such that*

$$0 < \underline{\alpha} \leq \alpha(y) \leq \overline{\alpha} < 2, \quad y \in \mathbf{R}^n.$$

Then, the generator A of stable-like process is given for $u \in C_0^2(\mathbf{R}^n)$ by

$$\begin{aligned} Au(y) &= -(-\Delta)^{\alpha(y)/2}u(y) \\ &= \int_{x\neq 0} \left(u(y+x) - u(y) - \sum_{i=1}^n x_i \frac{\partial u}{\partial y_i}(y) I_{\{|x|\leq 1\}}(x) \right) \frac{w(y)}{|x|^{n+\alpha(y)}} dx, \end{aligned}$$

where dx is the Lebesgue measure on $\mathbf{R}^n$ and the function w is given by

$$w(y) = \frac{\Gamma((1+\alpha(y))/2)\Gamma((n+\alpha(y))/2)\sin(\pi\alpha(y)/2)}{2^{1-\alpha(y)}\pi^{n/2+1}}, \quad y \in \mathbf{R}^n.$$

Further, Proposition 4.1 of Uemura (2014b) shows that the dual generator $\hat{A}$ of A on the space $L^2(\mathbf{R}^n; dx)$ has the following representation for $u \in C_0^2(\mathbf{R}^n)$:

$$\hat{A}u(y) = -\widehat{(-\Delta)^{\alpha(y)/2}}u(y)$$

$$
\begin{aligned}
= & \int_{x\neq 0}\left(u(y+x)-u(y)-\sum_{i=1}^{n}x_i\frac{\partial u}{\partial y_i}(y)I_{\{|x|\le 1\}}(x)\right)\frac{w(y+x)}{|x|^{n+\alpha(y+x)}}dx \\
& +\frac{1}{2}\int_{0<|x|\le 1}\sum_{i=1}^{n}x_i\frac{\partial u}{\partial y_i}(y)\left(\frac{w(y+x)}{|x|^{n+\alpha(y+x)}}-\frac{w(y)}{|x|^{n+\alpha(y)}}\right)dx \\
& -u(y)\int_{x\neq 0}\Bigg(w(y+x)|x|^{\overline{\alpha}-\alpha(y+x)}-w(y)|x|^{\overline{\alpha}-\alpha(y)} \\
& \qquad -\sum_{i=1}^{n}x_i\frac{\partial(w(y)|x|^{\overline{\alpha}-\alpha(y)})}{\partial y_i}I_{\{|x|\le 1\}}(x)\Bigg)\frac{dx}{|x|^{n+\overline{\alpha}}}.
\end{aligned}
$$

Example 2.15 (see Uemura (2014a)) *Let U be an open set of $\mathbf{R}^n$. Suppose that the following conditions hold:*

(C.I) There exist $0<\lambda\le\Lambda$ such that

$$
\lambda|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2 \quad \text{for } x\in U,\ \xi\in\mathbf{R}^n.
$$

(C.II) $b_i\in L^n(U;dx)$, $i=1,\dots,n$.

(C.III) $c\in L_+^{n/2}(U;dx)$.

(C.IV) $k_s(x,y)$ is a nonnegative measurable function defined on $U\times U\backslash d$ satisfying

$$
x\to\int_{y\neq x}(1\wedge|x-y|^2)k_s(x,y)dy\in L^1_{loc}(U;dx),
$$

(C.V) $k_a(x,y)$ is a measurable function defined on $U\times U\backslash d$ satisfying

$$
\sup_{x\in U}\int_{\{|x-y|\ge 1,y\in U\}}|k_a(x,y)|dy<\infty,
$$

$$
\sup_{x\in U}\int_{\{|x-y|<1,y\in U\}}|k_a(x,y)|^\gamma dy<\infty \text{ for some } 0<\gamma\le 1,
$$

and $|k_a(x,y)|^{2-\gamma}\le Ck_s(x,y)$, $x,y\in U$ with $0<|x-y|<1$ for some constant $C>0$.

Define for $u,v\in C_0^1(U)$,

$$
\begin{aligned}
\mathcal{Q}(u,v) &= \frac{1}{2}\sum_{i=1}^{n}\int_U\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)dx \\
&\qquad +\frac{1}{2}\iint_{x\neq y}(u(x)-u(y))(v(x)-v(y))k_s(x,y)dxdy
\end{aligned}
$$

and

$$
\mathcal{E}(u,v) = \frac{1}{2}\sum_{i,j=1}^{n}\int_U a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)dx+\sum_{i=1}^{n}\int_U b_i(x)u(x)\frac{\partial v}{\partial x_i}(x)dx
$$

$$
\begin{aligned}
&+\int_U u(x)v(x)c(x)dx \\
&+\frac{1}{2}\iint_{x\neq y}(u(x)-u(y))(v(x)-v(y))k_s(x,y)dxdy \\
&+\iint_{x\neq y}(u(x)-u(y))v(x)k_a(x,y)dxdy.
\end{aligned}
$$

Then, Uemura (2014a) Theorem 3.1 shows that when λ is sufficiently large, there exists a positive constant β such that $(\mathcal{E}_\beta, C_0^1(U))$ is closable on $L^2(U;dx)$ and its closure $(\mathcal{E}_\beta, D(\mathcal{E}_\beta))$ is a regular semi-Dirichlet form on $L^2(U;dx)$. Note that Assumption 2.12 (hence Assumption 2.3) is satisfied for $(\mathcal{E}_\beta, D(\mathcal{E}_\beta))$ since there exists a constant $C' > 1$ such that for any $u \in C_0^1(U)$,

$$\frac{1}{C'}\mathcal{Q}_1(u,u) \leq \mathcal{E}_\beta(u,u) \leq C'\mathcal{Q}_1(u,u).$$

2.3 LeJan type transformation rule for the diffusion part of regular semi-Dirichlet forms

In this section, we will apply some ideas of Section 2.1 to investigate the structure of general regular semi-Dirichlet forms. We will generalize the LeJan type transformation rule of Hu et al. (2010) to the semi-Dirichlet forms setting, see Theorems 2.17 and 2.20 below. Note that if $(\mathcal{E}, D(\mathcal{E}))$ is only a semi-Dirichlet form, its dual form $(\hat{\mathcal{E}}, D(\mathcal{E}))$ generally does not satisfy the semi-Dirichlet property. So we do not have the decomposition (1.1.5). In particular, the existence of the dual killing measure $\hat{K}$ is not ensured. Also, the symmetric part $\tilde{\mathcal{E}}$ of $\mathcal{E}$ is only a symmetric positivity preserving form but not a symmetric Dirichlet form, which causes extra difficulty in characterizing the structure of $\mathcal{E}$.

Throughout this section, we let E be a locally compact separable metric space, m be a positive Radon measure on E with $\mathrm{supp}[m] = E$, and $(\mathcal{E}, D(\mathcal{E}))$ a regular semi-Dirichlet form on $L^2(E;m)$.

We use J and K to denote respectively the jumping and killing measures of $(\mathcal{E}, D(\mathcal{E}))$. By Corollary 2.2 of Hu et al. (2006), there exists a unique positive Radon measure σ_β on

$E \times E$ satisfying

$$(\beta G_\beta u, v) = \int_{E\times E} u(x)v(y)\sigma_\beta(dxdy) \text{ for } u, v \in L^2(E; m).$$

Hereafter $(G_\beta)_{\beta>0}$ denotes the resolvent of $(\mathcal{E}, D(\mathcal{E}))$. We have $(\beta/2)\sigma_\beta \to J$ vaguely on $E \times E \backslash d$ as $\beta \to \infty$ (cf. the proof of Theorem 2.6 of Hu et al. (2006)). Define $\hat{J}(dxdy) := J(dydx)$, $\hat{\sigma}_\beta(dxdy) := \sigma_\beta(dydx)$, and denote by $(\hat{G}_\beta)_{\beta>0}$ the co-resolvent of $(\mathcal{E}, D(\mathcal{E}))$. Then, we have

$$(\beta \hat{G}_\beta u, v) = \int_{E\times E} u(x)v(y)\hat{\sigma}_\beta(dxdy) \text{ for } u, v \in L^2(E; m)$$

and $(\beta/2)\hat{\sigma}_\beta \to \hat{J}$ vaguely on $E \times E \backslash d$ as $\beta \to \infty$.

Let ρ be the metric on E. We choose a sequence of relatively compact open sets $\Omega_l \uparrow E$ and a sequence of numbers $\varsigma_l \downarrow 0$ such that the set $\Gamma_l = \{(x, y) \in \Omega_l \times \Omega_l : \rho(x, y) \geq \varsigma_l\}$ is a continuous set w.r.t. J for every $l \in \mathbf{N}$. Denote $\Lambda_l = \{(x, y) \in \Omega_l \times \Omega_l : \rho(x, y) < \varsigma_l\}$.

We make the following assumption.

Assumption 2.16 *For $f, g \in C_0(E) \cap D(\mathcal{E})$, we have $fg \in C_0(E) \cap D(\mathcal{E})$ and the function $(f(y) - f(x))g(y)$ is integrable w.r.t. J.*

Suppose $u, v \in C_0(E) \cap D(\mathcal{E})$. Let $\chi \in C_0(E) \cap D(\mathcal{E})$ satisfying $\chi = 1$ on a neighbourhood of $\mathrm{supp}[u] \cup \mathrm{supp}[v]$. Then, by Assumption 2.16, we get

$$\begin{aligned}
&\hat{\mathcal{E}}(u, v) = \lim_{\beta\to\infty} \beta(u - \beta\hat{G}_\beta u, v) \\
= \;& \lim_{\beta\to\infty} \beta \left\{ \int_{E\times E} (u(y) - u(x))v(y)\chi(x)\hat{\sigma}_\beta(dxdy) \right. \\
&\quad \left. + \int_E \chi(x)u(x)v(x)m(dx) - \int_{E\times E} \chi(x)u(y)v(y)\hat{\sigma}_\beta(dxdy) \right\} \\
= \;& \lim_{\beta\to\infty} \beta \int_{E\times E} (u(y) - u(x))v(y)\chi(x)\hat{\sigma}_\beta(dxdy) + \hat{\mathcal{E}}(\chi, uv) \\
= \;& \lim_{l\to\infty} \lim_{\beta\to\infty} \beta \int_{\Lambda_l} (u(y) - u(x))v(y)\hat{\sigma}_\beta(dxdy) \\
&+ \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)\chi(x)\hat{J}(dxdy) + \hat{\mathcal{E}}(\chi, uv). \qquad (2.3.1)
\end{aligned}$$

Hence we can well-define

$$\hat{\mathcal{E}}^c(u, v) := \lim_{l\to\infty} \lim_{\beta\to\infty} \beta \int_{\Lambda_l} (u(y) - u(x))v(y)\hat{\sigma}_\beta(dxdy). \qquad (2.3.2)$$

and observe that $\hat{\mathcal{E}}^c(u,v)$ satisfies the left strong local property in the sense that $\hat{\mathcal{E}}^c(u,v)=0$ whenever u is constant on a neighbourhood of supp$[v]$. By (2.3.1) and (2.3.2), we obtain the decomposition

$$\hat{\mathcal{E}}(u,v)=\hat{\mathcal{E}}^c(u,v)+\int_{E\times E\backslash d}2(u(y)-u(x))v(y)\chi(x)\hat{J}(dxdy)+\hat{\mathcal{E}}(\chi,uv). \tag{2.3.3}$$

Similar to (2.3.1), we can show that

$$\begin{aligned}\mathcal{E}(u,v) &= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(u(y)-u(x))v(y)\sigma_\beta(dxdy)\\ &\quad +\int_{E\times E\backslash d}2(u(y)-u(x))v(y)\chi(x)J(dxdy)+\mathcal{E}(\chi,uv).\end{aligned} \tag{2.3.4}$$

By (1.1.4) and (2.3.4), we get

$$\begin{aligned}\mathcal{E}^c(u,v) &= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(u(y)-u(x))v(y)\sigma_\beta(dxdy)\\ &\quad +\int_{E\times E\backslash d}2(u(y)-u(x))v(y)\chi(x)J(dxdy)\\ &\quad +\int_{E\times E\backslash d}2(\chi(y)-\chi(x))(uv)(y)J(dxdy)+\int_E u(x)v(x)K(dx)\\ &\quad -\int_{E\times E\backslash d}2(u(y)-u(x))v(y)J(dxdy)-\int_E u(x)v(x)K(dx)\\ &= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(u(y)-u(x))v(y)\sigma_\beta(dxdy).\end{aligned} \tag{2.3.5}$$

For $r\in\mathbf{N}$, we choose a $w\in C_0(E)\cap D(\mathcal{E})$ satisfying $w\geq 0$ and $w|_{\Omega_r}\equiv 1$. For $f\in C_0(\Omega_r)\cap D(\mathcal{E})$, we obtain by (2.3.5) and the sub-Markovian property of $(G_\beta)_{\beta>0}$ that

$$\begin{aligned}&|2\mathcal{E}^c(u,uf)-\mathcal{E}^c(u^2,f)|\\ =\ &\left|\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\int_{\Lambda_l}(u(y)-u(x))^2f(y)\sigma_\beta(dxdy)\right|\\ \leq\ &\|f\|_\infty\lim_{\beta\to\infty}\beta\int_{E\times E}(u(y)-u(x))^2w(y)\sigma_\beta(dxdy)\\ \leq\ &\|f\|_\infty\lim_{\beta\to\infty}\{2\beta(u-\beta G_\beta u,uw)-\beta(u^2-\beta G_\beta u^2,w)\}\\ =\ &(2\mathcal{E}(u,uw)-\mathcal{E}(u^2,w))\|f\|_\infty.\end{aligned}$$

Then, there exists a unique Radon measure $\mu^{r,c}_{<u>}$ on Ω_r such that

$$\int_{\Omega_r}fd\mu^{r,c}_{<u>}=2\mathcal{E}^c(u,uf)-\mathcal{E}^c(u^2,f),\quad \forall f\in C_0(\Omega_r)\cap D(\mathcal{E}).$$

It is easy to see that $\{\mu^{r,c}_{<u>}\}$ is a consistent sequence of Radon measures. Therefore, we can well define the measure $\mu^c_{<u>}$ by $\mu^c_{<u>} = \mu^{r,c}_{<u>}$ on Ω_r, which satisfies

$$\int_E f d\mu^c_{<u>} = 2\mathcal{E}^c(u, uf) - \mathcal{E}^c(u^2, f), \quad \forall f \in C_0(E) \cap D(\mathcal{E}).$$

We define

$$\mu^c_{<u,v>} := \frac{1}{2}(\mu^c_{<u+v>} - \mu^c_{<u>} - \mu^c_{<v>}).$$

Then

$$\int_E f d\mu^c_{<u,v>} = \mathcal{E}^c(u, vf) + \mathcal{E}^c(v, uf) - \mathcal{E}^c(uv, f), \quad f \in C_0(E) \cap D(\mathcal{E}).$$

Hence, for any $h \in C_0(E) \cap D(\mathcal{E})$ satisfying $h|_{\mathrm{supp}[u] \cup \mathrm{supp}[v]} \equiv 1$, we have

$$\mathcal{E}^c(u, v) + \mathcal{E}^c(v, u) = \int_E h d\mu^c_{<u,v>} + \mathcal{E}^c(uv, h). \tag{2.3.6}$$

We define a linear functional $L(u, v)$ on $C_0(E) \cap D(\mathcal{E})$ by

$$< L(u, v), f >:= \frac{1}{2}(\mathcal{E}^c(u, vf) - \hat{\mathcal{E}}^c(u, vf)), \quad f \in C_0(E) \cap D(\mathcal{E}).$$

Then, for any $h \in C_0(E) \cap D(\mathcal{E})$ satisfying $h|_{\mathrm{supp}[u]} \equiv 1$, we have

$$\mathcal{E}^c(u, v) - \hat{\mathcal{E}}^c(u, v) = 2 < L(u, v), h > . \tag{2.3.7}$$

By (1.1.4), (2.3.3) and (2.3.7), we get

$$\begin{aligned}
& \mathcal{E}^c(u, v) - \mathcal{E}^c(v, u) \\
= \ & \mathcal{E}^c(u, v) - \mathcal{E}(v, u) + \int_E u(x)v(x)K(dx) + \int_{E \times E \backslash d} 2(v(y) - v(x))u(y)J(dxdy) \\
= \ & \mathcal{E}^c(u, v) - \hat{\mathcal{E}}^c(u, v) - \hat{\mathcal{E}}(\chi, uv) + \int_E u(x)v(x)K(dx) \\
& - \int_{E \times E \backslash d} 2(u(y) - u(x))v(y)\chi(x)\hat{J}(dxdy) + \int_{E \times E \backslash d} 2(v(y) - v(x))u(y)J(dxdy) \\
= \ & 2 < L(u, v), \chi > -\mathcal{E}(uv, \chi) + \int_U u(x)v(x)K(dx) \\
& - \int_{E \times E \backslash d} 2(u(y) - u(x))v(y)\chi(x)\hat{J}(dxdy) + \int_{E \times E \backslash d} 2(v(y) - v(x))u(y)J(dxdy) \\
= \ & 2 < L(u, v), \chi > -\mathcal{E}^c(uv, \chi) \\
& - \int_{E \times E \backslash d} 2((uv)(y) - (uv)(x))\chi(y)J(dxdy)
\end{aligned}$$

$$-\int_{E\times E\backslash d} 2(u(x)-u(y))v(x)\chi(y)J(dxdy)+\int_{E\times E\backslash d} 2(v(y)-v(x))u(y)J(dxdy)$$
$$= \quad 2 < L(u,v),\chi > -\mathcal{E}^c(uv,\chi). \tag{2.3.8}$$

By (2.3.6) and (2.3.8), we obtain the following expression of the diffusion part $\mathcal{E}^c$.

Theorem 2.17 *Suppose Assumption 2.16 holds. Let $u, v \in C_0(E)\cap D(\mathcal{E})$ and let $\chi \in C_0(E)\cap D(\mathcal{E})$ satisfying $\chi = 1$ on a neighbourhood of* $\mathrm{supp}[u]\cup\mathrm{supp}[v]$. *Then*

$$\mathcal{E}^c(u,v)=\frac{1}{2}\int_E \chi d\mu^c_{<u,v>}+ < L(u,v),\chi > .$$

Similar to Theorem 2.9, we can derive the following transformation rules for $\mu^c_{<\cdot,\cdot>}$ and $L(\cdot,\cdot)$.

Theorem 2.18 *Let $u, v, w, f \in C_0(E)\cap D(\mathcal{E})$. Then*
(i) $d\mu^c_{<uv,w>} = ud\mu^c_{<v,w>} + vd\mu^c_{<u,w>}$.
(ii) $< L(u, vw), f >=< L(u, v), wf >$.
(iii) $< L(uv, w), f >=< L(u, w), vf > + < L(v, w), uf >$.

We use $\mathcal{F}_{loc}$ to denote the set of all functions u such that for any relatively compact open set V there exists a $w \in C_0(E)\cap D(\mathcal{E})$ such that $u = w$ on V. Then, by an argument similar to that given after the proof of Theorem 2.9, we can extend $\mu^c_{<u,v>}$ and $L(u, v)$ to $u, v \in \mathcal{F}_{loc}$. The transformation rules given in Theorem 2.18 still hold with $C_0(E)\cap D(\mathcal{E})$ replaced by $\mathcal{F}_{loc}$.

Now we make the following assumption.

Assumption 2.19 *There exist a sequence of Dirichlet forms $(\mathcal{Q}^l, D(\mathcal{Q}^l))$ on the space $L^2(\Omega_l; m)$ and a sequence of positive constants C_l such that*

$$C_0(\Omega_l)\cap D(\mathcal{E}) = C_0(\Omega_l)\cap D(\mathcal{Q}_l)$$

and

$$\mathcal{E}_1(g, g) \le C_l\mathcal{Q}^l_1(g, g), \quad \forall g \in C_0(\Omega_l)\cap D(\mathcal{E}).$$

Theorem 2.20 *Suppose Assumption 2.19 holds and suppose J is a finite measure on $E \times E \backslash d$. Let $\Phi \in C^2(\mathbf{R}^m)$, $u_1, \dots, u_m, v, w \in \mathcal{F}_{loc}$ and $f \in C_0(E) \cap D(\mathcal{E})$. Then*
(i) $d\mu^c_{<\Phi(u_1,\dots,u_m),v>} = \sum_{i=1}^m \Phi_{x_i}(u_1, \dots, u_m) d\mu^c_{<u_i,v>}$.
(ii) $< L(\Phi(u_1, \dots, u_m), vw), f >= \sum_{i=1}^m < L(u_i, v), \Phi_{x_i}(u_1, \dots, u_m) wf >$.

The proof of Theorem 2.20 is similar and simpler than that of Theorem 2.10 above. Therefore, we omit the details here. We only point out that Fukushima et al. (2011) (3.2.27) and Assumption 2.19 ensure the convergence of $\Phi^{(k)}(u)$ (resp. $\Phi^{(k)}_{x_i}(u) - \Phi^{(k)}_{x_i}(0)$) to $\Phi(u)$ (resp. $\Phi_{x_i}(u) - \Phi_{x_i}(0)$) w.r.t. the $\tilde{\mathcal{Q}}_1^{1/2}$-norm and hence the $\tilde{\mathcal{E}}_1^{1/2}$-norm, and the finiteness of J ensures that the dominated convergence theorem can be applied directly.

Chapter 3

Probabilistic representations of solutions of elliptic boundary value problem and non-symmetric semigroups

In this chapter, we first use probabilistic methods to study the Dirichlet boundary value problem (1.1.6) for the following second order elliptic differential operators L:

$$Lu = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + (c(x) - \operatorname{div}\hat{b}(x))u. \tag{3.0.1}$$

Then, we will use similar techniques to give a probabilistic representation of the non-symmetric semigroup $\{T_t\}_{t\geq 0}$ associated with the same operator L.

3.1 Probabilistic representation of solutions of elliptic boundary value problem

In this section, we discuss the Dirichlet boundary value problem (1.1.6) for the operator (3.0.1), where $A(x) = (a_{ij}(x))_{i,j=1}^n$ is a Borel measurable, (not necessarily symmetric) matrixvalued function on D satisfying

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \text{for any } \xi = (\xi_i)_{i=1}^n \in \mathbf{R}^n, x \in D \tag{3.1.1}$$

and

$$|a_{ij}(x)| \leq \frac{1}{\lambda} \quad \text{for any } x \in D, \ 1 \leq i, j \leq n \tag{3.1.2}$$

for some constant $0 < \lambda \leq 1$; $b = (b_1, \ldots, b_n)^*$ and $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_n)^*$ are Borel measurable $\mathbf{R}^n$-valued functions on D and c is a Borel measurable function on D satisfying $|b|^2 \in L^{p\vee 1}(D; dx)$, $|\hat{b}|^2 \in L^{p\vee 1}(D; dx)$ and $c \in L^{p\vee 1}(D; dx)$ for some constant $p > n/2$.

The bilinear form $(\mathcal{E}, D(\mathcal{E}))$ associated with L is:

$$\begin{aligned} \mathcal{E}(u, v) &= \frac{1}{2}\sum_{i,j=1}^{n} \int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_{i=1}^{n}\int_D b_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ &\quad - \sum_{i=1}^{n}\int_D \hat{b}_i(x)\frac{\partial(uv)}{\partial x_i}dx - \int_D c(x)u(x)v(x)dx, \\ D(\mathcal{E}) &= H_0^{1,2}(D) \end{aligned} \tag{3.1.3}$$

with $H_0^{1,2}(D)$ being the completion of $C_0^\infty(D)$ with respect to the Sobolev-norm $\|\cdot\|_{H^{1,2}}$.

By setting $a = I$, $b = 0$, $\hat{b} = 0$ and $c = 0$ off D, we may assume that the operator L is defined on $\mathbf{R}^n$.

Note that the operator L given by (3.0.1) is the same as that used in Chen and Zhang (2009) (i.e.,(1.1.13)) if we replace b with $b - \hat{b}$ in (3.0.1). The results of Chen and Zhang (2009) are based on the Condition (1.1.15). If Condition (1.1.15) is replaced with other conditions which guarantee that the gradient ∇u in (1.1.16) belongs to some L^p space for $p > n$, e.g. the condition that A is in the class VMO and $\partial D \in C^{1,1}$ (see Di Fazio (1996)), then Chen and Zhang's approach still apply.

In general, it is possible that $f \in L^p$ while $\nabla u \notin L^p$ (see Meyers (1963) for an example). For this case, we cannot use the h-transform method to tackle the lower-order term $\operatorname{div}\hat{b}$ even when A is symmetric. In this section, we will show that there exists a unique, bounded continuous solution to Dirichlet boundary value problem for the operator (3.0.1) without additional condition on A such as Condition (1.1.15), the VMO condition or the symmetry of A, and without the Markovian assumption (1.1.14). Instead of using Meyers's L^p-estimate as in Chen and Zhang (2009), we will make use of Aronson's heat kernel estimates(cf. Aronson (1967, 1968)).

In the sequel, we let $X = ((X_t)_{t\geq 0}, (P_x)_{x\in\mathbf{R}^n})$ be the Markov process associated with the following (non-symmetric) Dirichlet form

$$\begin{aligned} \mathcal{E}^0(u,v) &= \frac{1}{2}\int_{\mathbf{R}^n}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx, \\ D(\mathcal{E}^0) &= H^{1,2}(\mathbf{R}^n). \end{aligned} \tag{3.1.4}$$

It is well-known that X is a conservative Feller process on $\mathbf{R}^n$ that has continuous transition density function which admits a two-sided Aronson's heat kernel estimate. Let $\{\mathcal{F}_t, t\geq 0\}$ be the minimal augmented filtration generated by X. By Fukushima's decomposition (cf. Oshima (1988) Theorem 5.1.8), we have

$$X_t = x + M_t + N_t,$$

where $M_t = (M_t^1,\dots,M_t^n)^*$ is a martingale additive functional of X with quadratic covariation

$$\langle M^i, M^j\rangle_t = \int_0^t \tilde{a}_{ij}(X_s)ds$$

and $N_t = (N_t^1,\dots,N_t^n)^*$ is a continuous additive functional of X locally of zero quadratic variation. Hereafter $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^n$ denotes the symmetrization of A, i.e., $\tilde{A} := 1/2(A + A^*)$.

For any vector field $\xi \in L^2(\mathbf{R}^n; dx)$, there exists a unique function $\xi^H \in H^{1,2}(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n}\langle \xi, \nabla h\rangle dx = -\mathcal{E}_1^0(\xi^H, h), \quad \forall h\in C_0^\infty(\mathbf{R}^n)$$

(see Lemma 3.3 below). We have Fukushima's decomposition:

$$\widetilde{\xi^H}(X_t) - \widetilde{\xi^H}(X_0) = M_t^{\xi^H} + N_t^{\xi^H},$$

where $\widetilde{\xi^H}$ is a quasi-continuous version of ξ^H. To simplify notation, in the sequel we take w to be its quasi-continuous version $\tilde{w}$ whenever such a version exists. As in Fukushima et al.

(2011) and Ma and Röckner (1992), we use the term “quasi-everywhere” (abbreviated “q.e.”) to mean “except on an exceptional set”.

Now we can state the main theorem of this section.

Theorem 3.1 *Let $n \geq 1$, D be a bounded Lipschitz domain in $\mathbf{R}^n$ and $p > n/2$. Suppose that*

(i) A satisfies (3.1.1) and (3.1.2).

(ii) $|b|^2 \in L^{p\vee 1}(D; dx)$ and $|\hat{b}|^2 \in L^{p\vee 1}(D; dx)$.

(iii) $c \in L^{p\vee 1}(D; dx)$ and $c - \operatorname{div} \hat{b} \leq g$ for some nonnegative function $g \in L^{p\vee 1}(D; dx)$ in the distributional sense.

Then, there exists a constant $M > 0$ such that whenever $\|g\|_{L^{p\vee 1}} \leq M$, for any $f \in C(\partial D)$, there exists a unique weak solution u to $Lu = 0$ in D that is continuous on $\overline{D}$ with $u = f$ on ∂D. Moreover, the solution u admits the following representation for q.e. $x \in D$,

$$\begin{aligned} u(x) \;=\; & E_x\Bigg[\exp\bigg(\int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^{\tau_D} b^*\tilde{a}^{-1}b(X_s)ds \\ & + \int_0^{\tau_D} c(X_s)ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D} \hat{b}^H(X_s)ds\bigg) f(X_{\tau_D})\Bigg]. \end{aligned} \tag{3.1.5}$$

We will give the proof of Theorem 3.1 in Section 3.2, which consists of three subsections. In Subsection 3.2.1, we prove the existence of the weak solution and give its probabilistic representation (3.1.5). In Subsection 3.2.2, we prove the continuity of the weak solution. In Subsection 3.2.3, we prove the uniqueness of the continuous weak solutions. The recently developed Nakao integral for non-symmetric Dirichlet forms (cf. Chen et al. (2012) and Walsh (2013)) will be used in the proof of the uniqueness.

3.2 Proof of Theorem 3.1

3.2.1 Proof of the existence of weak solution

We first generalize Theorem 1.1 of Chen and Zhao (1995) from the case of symmetric diffusion matrix A to the non-symmetric case. Define

$$L^1 u = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

Lemma 3.2 *Suppose that D is a bounded domain in $\mathbf{R}^n$, $c \le 0$ and $f \in C(\partial D)$. Then*

$$u(x) = E_x\left[\exp\left(\int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^{\tau_D} b^*\tilde{a}^{-1}b(X_s)ds + \int_0^{\tau_D} c(X_s)ds\right) f(X_{\tau_D})\right]$$

is the unique weak solution of $L^1 u = 0$ which is continuous in D and

$$\lim_{x\to y, x\in D} u(x) = f(y)$$

for $y \in \partial D$ which is regular for the Laplace operator $(\frac{1}{2}\triangle, D)$.

Proof. The proof of Lemma 3.2 is similar to that of Theorem 1.1 of Chen and Zhao (1995). We only point out below the main differences in the argument between the symmetric and the non-symmetric cases.

Denote by X^0 the part of the process X on D, that is, X^0 is obtained by killing the sample paths of X upon leaving D. By Aronson (1967, 1968), the transition density function $p_0(t,x,y)$ of X^0 has the upbound estimate

$$p_0(t,x,y) \le \frac{\vartheta}{t^{n/2}} e^{-\frac{|x-y|^2}{\vartheta t}}, \quad (t,x,y) \in (0,\infty)\times D \times D, \tag{3.2.1}$$

for some constant $\vartheta > 0$.

We define

$$L^0 u = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right).$$

Let D_1 be a bounded subdomain of D and $f_1 \in H_0^{1,2}(D)$. By Trudinger (1973), there exists a unique weak solution of $L^0 u = 0$ in D_1 such that $u - f_1|_{D_1} \in H_0^{1,2}(D_1)$. Further, by the famous theorem of Littman, Stampacchia and Weinberger, which holds also for the non-symmetric case (cf. e.g. Kenig et al. (2000)), we can prove the analog of Theorem 2.1 of Chen and Zhao (1995) with the non-symmetric A. By virtue of the Harnack inequality for parabolic equations (cf. Moser (1964) and Lierl and Saloff-Coste (2012)), we can prove that Lemma 2.2 of Chen and Zhao (1995) and hence Corollary 2.3 and Theorem 2.4 of Chen and Zhao (1995) hold for the non-symmetric case.

Finally, we would like to point out that the exponential martingale M_t introduced in Chen and Zhao (1995) needs to be replaced with

$$U_t := \exp\left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds\right), \quad t \geq 0 \tag{3.2.2}$$

for our non-symmetric case. □

Lemma 3.3 *(i) For any vector field* $\xi \in L^2(\mathbf{R}^n; dx)$*, there exists a unique function* $\xi^H \in H^{1,2}(\mathbf{R}^n)$ *such that*

$$\int_{\mathbf{R}^n} \langle \xi, \nabla h \rangle dx = -\mathcal{E}_1^0(\xi^H, h), \quad \forall h \in H^{1,2}(\mathbf{R}^n). \tag{3.2.3}$$

(ii) If ξ_n *converges to* ξ *in* $L^2(\mathbf{R}^n; dx)$ *as* $n \to \infty$*, then* ξ_n^H *converges to* ξ^H *in* $H^{1,2}(\mathbf{R}^n)$ *as* $n \to \infty$.

(iii) For $\xi \in C_0^\infty(\mathbf{R}^n)$,

$$-\int_0^t \operatorname{div}\xi(X_s)ds = N_t^{\xi^H} - \int_0^t \xi^H(X_s)ds, \quad t \geq 0. \tag{3.2.4}$$

Proof. (i) Let $\xi \in L^2(\mathbf{R}^n; dx)$. We define the map $\eta : h \in H^{1,2}(\mathbf{R}^n) \mapsto \int_{\mathbf{R}^n} \langle \xi, \nabla h \rangle dx$. By the Riesz representation theorem, there exists a unique $\xi^0 \in H^{1,2}(\mathbf{R}^n)$ such that

$$\eta(h) = \tilde{\mathcal{E}}_1^0(\xi^0, h), \quad \forall h \in H^{1,2}(\mathbf{R}^n), \tag{3.2.5}$$

where $(\tilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ denotes the symmetric part of the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$. Thus, by Lemma 2.1 of Chen et al. (2012), there exists a unique $\xi^H \in D(\mathcal{E}^0) = H^{1,2}(\mathbf{R}^n)$ such that

$$\tilde{\mathcal{E}}_1^0(\xi^0, h) = -\mathcal{E}_1^0(\xi^H, h), \quad \forall h \in H^{1,2}(\mathbf{R}^n). \tag{3.2.6}$$

(ii) Suppose ξ_n converges to ξ in $L^2(\mathbf{R}^n;dx)$ as $n\to\infty$. By (3.2.5), we get

$$
\begin{aligned}
\|\xi_n^0-\xi^0\|_{\tilde{\mathcal{E}}_1^0} &= \sup_{\|h\|_{\tilde{\mathcal{E}}_1^0}=1}\tilde{\mathcal{E}}_1^0(\xi_n^0-\xi^0,h)\\
&= \sup_{\|h\|_{\tilde{\mathcal{E}}_1^0}=1}\int_{\mathbf{R}^n}\langle\xi_n-\xi,\nabla h\rangle dx\\
&\le \|\xi_n-\xi\|_{L^2}\sup_{\|h\|_{\tilde{\mathcal{E}}_1^0}=1}\|h\|_{H^{1,2}}\\
&\to 0 \text{ as } n\to\infty. \qquad (3.2.7)
\end{aligned}
$$

Further, by (3.2.6), we get

$$
\begin{aligned}
\mathcal{E}_1^0(\xi_n^H-\xi^H,\xi_n^H) &= \mathcal{E}_1^0(\xi_n^H,\xi_n^H)-\mathcal{E}_1^0(\xi^H,\xi_n^H)\\
&= -\tilde{\mathcal{E}}_1^0(\xi_n^0,\xi_n^H)+\tilde{\mathcal{E}}_1^0(\xi^0,\xi_n^H)\\
&= \tilde{\mathcal{E}}_1^0(\xi^0-\xi_n^0,\xi_n^H)\\
&\le \left[\tilde{\mathcal{E}}_1^0(\xi^0-\xi_n^0,\xi^0-\xi_n^0)\right]^{1/2}\left[\tilde{\mathcal{E}}_1^0(\xi_n^H,\xi_n^H)\right]^{1/2}, \qquad (3.2.8)
\end{aligned}
$$

$$
\sup_{n\in\mathbf{N}}\mathcal{E}_1^0(\xi_n^H,\xi_n^H)\le\sup_{n\in\mathbf{N}}\tilde{\mathcal{E}}_1^0(\xi_n^0,\xi_n^0)<\infty, \qquad (3.2.9)
$$

and

$$
\begin{aligned}
\lim_{n\to\infty}\mathcal{E}_1^0(\xi_n^H-\xi^H,\xi^H) &= -\lim_{n\to\infty}\tilde{\mathcal{E}}_1^0(\xi_n^0-\xi^0,\xi^H)\\
&= -\lim_{n\to\infty}\int_{\mathbf{R}^n}\langle\xi_n-\xi,\nabla\xi^H\rangle dx\\
&= 0. \qquad (3.2.10)
\end{aligned}
$$

Therefore, we obtain by (3.2.7)-(3.2.10) that

$$
\begin{aligned}
\lim_{n\to\infty}\mathcal{E}_1^0(\xi_n^H-\xi^H,\xi_n^H-\xi^H) &= \lim_{n\to\infty}\left\{\mathcal{E}_1^0(\xi_n^H-\xi^H,\xi_n^H)-\mathcal{E}_1^0(\xi_n^H-\xi^H,\xi^H)\right\}\\
&= 0.
\end{aligned}
$$

(iii) Let $\xi\in C_0^\infty(\mathbf{R}^n)$. For any $h\in H^{1,2}(\mathbf{R}^n)$, we have

$$
\begin{aligned}
\lim_{t\to 0}\frac{1}{t}E_{h\cdot dx}\left[-\int_0^t\operatorname{div}\xi(X_s)ds\right] &= -\int_{\mathbf{R}^n}(\operatorname{div}\xi)h\,dx\\
&= \int_{\mathbf{R}^n}\langle\xi,\nabla h\rangle dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\mathcal{E}_1^0(\xi^H, h) \\
&= \lim_{t\to 0}\frac{1}{t}E_{h\cdot dx}\left[N_t^{\xi^H} - \int_0^t \xi^H(X_s)ds\right].
\end{aligned}
$$

Therefore, (3.2.4) holds by Lemma 2.3 of Chen et al. (2012). □

Proof of the existence of weak solution and its probabilistic representation.

We define a family of measures $\{Q_x, x \in \mathbf{R}^n\}$ on $\mathcal{F}_\infty$ by

$$
\left.\frac{dQ_x}{dP_x}\right|_{\mathcal{F}_t} = U_t, \quad t \geq 0,
$$

where U_t is given by (3.2.2). Then, under $\{Q_x, x \in \mathbf{R}^n\}$, X is a diffusion process on $\mathbf{R}^n$ with the generator L^b given by

$$
L^b u = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}, \tag{3.2.11}
$$

Denote by E_x^Q the expectation with respect to the measure Q_x for $x \in \mathbf{R}^n$. **From now on till the end of this section, we fix a constant** $0 < \theta < \frac{1}{2}$. We will show below that there exists a constant $M > 0$ such that for any $w \in L^{p\vee 1}(\mathbf{R}^n; dx)$ with $\|w\|_{L^{p\vee 1}} \leq M$, we have

$$
\sup_{x\in D} E_x^Q\left[\int_0^{\tau_D} |w|(X_s)ds\right] \leq \theta. \tag{3.2.12}
$$

We only prove (3.2.12) when $n \geq 3$. The cases that $n = 1, 2$ can be considered similarly. Let X^D be the part of the process X on D under $\{Q_x\}$, that is, X^D is obtained by killing the sample paths of X upon leaving D. Denote by $p(t, x, y)$ the transition density function of X^D. By Theorem 9 of Aronson (1968), for each $T > 0$, there exist positive constants σ_1^T and σ_2^T such that

$$
p(t, x, y) \leq \frac{\sigma_1^T}{t^{n/2}}e^{-\frac{\sigma_2^T|x-y|^2}{t}}, \quad (t, x, y) \in (0, T) \times D \times D.
$$

Similar to the proof of Lemma 6.1 of Kim and Song (2006), we can show that there exist positive constants σ_1 and σ_2 such that

$$
p(t, x, y) \leq \frac{\sigma_1}{t^{n/2}}e^{-\frac{\sigma_2|x-y|^2}{t}}, \quad (t, x, y) \in (0, \infty) \times D \times D. \tag{3.2.13}
$$

Denote by $G_D(x,y)$ the Green function of X^D. Then,

$$G_D(x,y) \le \frac{\sigma_3}{|x-y|^{n-2}}, \quad (x,y) \in D \times D, \tag{3.2.14}$$

for some positive constant σ_3.

Let $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then $n - q(n-2) > 0$. We obtain by (3.2.14) that

$$\begin{aligned}
E_x^Q\left[\int_0^{\tau_D} |w|(X_s)ds\right] &= \int_D G_D(x,y)|w|(y)dy \\
&\le \int_D \frac{\sigma_3|w|(y)}{|x-y|^{n-2}}dy \\
&\le \sigma_3\left(\int_D (|w|(y))^p dy\right)^{1/p}\left(\int_D |x-y|^{-q(n-2)}dy\right)^{1/q} \\
&\le \sigma_3\|w\|_{L^p}\left(\int_0^{\varsigma} r^{n-q(n-2)-1}dr\right)^{1/q} \\
&= \frac{\sigma_3\varsigma^{n/q-(n-2)}}{[n-q(n-2)]^{1/q}}\|w\|_{L^p}.
\end{aligned}$$

Hereafter ς denotes the diameter of D. Set

$$M := \frac{\theta[n-q(n-2)]^{1/q}}{\sigma_3\varsigma^{n/q-(n-2)}}.$$

Then $\|w\|_{L^p} \le M$ implies (3.2.12). Further, by (3.2.12) and Khasminskii's inequality, we get

$$\sup_{x\in D} E_x^Q\left[\exp\left(\int_0^{\tau_D} |w|(X_s)ds\right)\right] \le \frac{1}{1-\theta}. \tag{3.2.15}$$

We define

$$J(x) = \frac{1_{\{|x|<1\}}e^{-\frac{1}{1-|x|^2}}}{\int_{\{|y|<1\}} e^{-\frac{1}{1-|y|^2}}dy}, \quad x \in \mathbf{R}^n.$$

For $k \in \mathbf{N}$ and $x \in \mathbf{R}^n$, set

$$\begin{aligned}
J_k(x) &:= k^n J(kx), \\
\hat{b}_k(x) &:= \int_{\mathbf{R}^n} \hat{b}(x-y)J_k(y)dy, \\
c_k(x) &:= \int_{\mathbf{R}^n} c(x-y)J_k(y)dy, \\
g_k(x) &:= \int_{\mathbf{R}^n} g(x-y)J_k(y)dy.
\end{aligned}$$

We have

$$\hat{b}_k \to \hat{b} \;\text{ in }\; L^2(\mathbf{R}^n; dx) \;\text{ as }\; k \to \infty \tag{3.2.16}$$

and

$$c_k \to c \;\text{ in }\; L^1(\mathbf{R}^n; dx) \;\text{ as }\; k \to \infty. \tag{3.2.17}$$

Suppose $\|g\|_{L^{p\vee 1}} \leq M$. Since $c - \operatorname{div}\hat{b} \leq g$ implies that $c_k - \operatorname{div}\hat{b}_k \leq g_k$ for $k \in \mathbf{N}$, we obtain by (3.2.15) that

$$\sup_{k\in\mathbf{N}}\sup_{x\in D} E_x^Q\left[\exp\left(\int_0^{\tau_D}(c_k - \operatorname{div}\hat{b}_k)(X_s)ds\right)\right] \leq \frac{1}{1-\theta}. \tag{3.2.18}$$

Define for $t \geq 0$,

$$\begin{aligned} Z_t \;&:=\; \exp\left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds\right. \\ &\qquad \left. + \int_0^t c(X_s)ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s)ds\right). \end{aligned} \tag{3.2.19}$$

By (3.2.16) and Lemma 3.3(ii), we get

$$\hat{b}_k^H \to \hat{b}^H \;\text{ in }\; H^{1,2}(\mathbf{R}^n) \;\text{ as }\; k \to \infty. \tag{3.2.20}$$

Further, by Lemma 4.1.12 and Theorem 5.1.2 of Oshima (1988), there exists a subsequence $\{k_l\}$ such that for q.e. $x \in \mathbf{R}^n$,

$$P_x\left\{\lim_{l\to\infty} N_t^{\hat{b}_{k_l}^H} = N_t^{\hat{b}^H} \text{ uniformly on any finite interval of } t\right\} = 1. \tag{3.2.21}$$

For simplicity, we still use $\{k\}$ to denote the subsequence $\{k_l\}$. By (3.2.17)-(3.2.21) and Fatou's lemma, we obtain that

$$\begin{aligned} E_x[Z_{\tau_D}] \;&=\; E_x^Q\left[\exp\left(\int_0^{\tau_D} c(X_s)ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D}\hat{b}^H(X_s)ds\right)\right] \\ &\leq\; \liminf_{k\to\infty} E_x^Q\left[\exp\left(\int_0^{\tau_D} c_k(X_s)ds + N_{\tau_D}^{\hat{b}_k^H} - \int_0^{\tau_D}\hat{b}_k^H(X_s)ds\right)\right] \\ &=\; \liminf_{k\to\infty} E_x^Q\left[\exp\left(\int_0^{\tau_D}(c_k - \operatorname{div}\hat{b}_k)(X_s)ds\right)\right] \\ &\leq\; \frac{1}{1-\theta}, \text{ for q.e. } x \in D. \end{aligned} \tag{3.2.22}$$

For $k \in \mathbf{N}$, we define

$$L_k u = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + (c_k(x) - \operatorname{div}\hat{b}_k(x))u.$$

The bilinear form $(\mathcal{E}_k, D(\mathcal{E}_k))$ associated with L_k is

$$\begin{aligned}
\mathcal{E}_k(u,v) &= \frac{1}{2}\sum_{i,j=1}^n \int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_{i=1}^n \int_D b_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\
&\quad - \sum_{i=1}^n \int_D \hat{b}_{k,i}(x)\frac{\partial (uv)}{\partial x_i}dx - \int_D c_k(x)u(x)v(x)dx, \\
D(\mathcal{E}_k) &= H_0^{1,2}(D).
\end{aligned}$$

By (3.2.18), following the argument of Theorem 4.3 of Chen and Zhang (2009), we can show that the weak solution to the Dirichlet boundary value problem

$$\begin{cases} L_k u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases} \tag{3.2.23}$$

is given by

$$\begin{aligned}
u_k(x) &= E_x^Q\left[\exp\left(\int_0^{\tau_D}(c_k - \operatorname{div}\hat{b}_k)(X_s)ds\right)f(X_{\tau_D})\right] \\
&= E_x\left[\exp\left(\int_0^{\tau_D}(\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^{\tau_D} b^*\tilde{a}^{-1}b(X_s)ds \right.\right. \\
&\qquad \left.\left. + \int_0^{\tau_D}(c_k - \operatorname{div}\hat{b}_k)(X_s)ds\right)f(X_{\tau_D})\right].
\end{aligned}$$

Denote by v the right-hand side of (3.1.5). We claim that

$$\lim_{k\to\infty} u_k(x) = v(x), \quad \text{for q.e. } x \in D. \tag{3.2.24}$$

In fact, define

$$\begin{aligned}
W_k &:= \exp\left(\int_0^{\tau_D}(c_k - \operatorname{div}\hat{b}_k)(X_s)ds\right) \\
&= \exp\left(\int_0^{\tau_D} c_k(X_s)ds + N_{\tau_D}^{\hat{b}_k^H} - \int_0^{\tau_D}\hat{b}_k^H(X_s)ds\right), \ k \in \mathbf{N}, \\
W &:= \exp\left(\int_0^{\tau_D} c(X_s)ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D}\hat{b}^H(X_s)ds\right).
\end{aligned}$$

By (3.2.17), (3.2.20) and (3.2.21), we get $W_k \to W$ in probability under Q_x as $k \to \infty$ for q.e. $x \in D$. By (3.2.12) and Khasminskii's inequality, we obtain that for $x \in D$,

$$
\begin{aligned}
\sup_{k\in\mathbf{N}} E_x^Q[W_k^2] &= \sup_{k\in\mathbf{N}} E_x^Q\left[\exp\left(2\int_0^{\tau_D}(c_k - \operatorname{div}\hat{b}_k)(X_s)ds\right)\right]\\
&\le \sup_{k\in\mathbf{N}} E_x^Q\left[\exp\left(2\int_0^{\tau_D} g_k(X_s)ds\right)\right]\\
&\le \frac{1}{1-2\theta}. \qquad (3.2.25)
\end{aligned}
$$

Hence $\{W_k\}$ is uniformly integrable under Q_x for $x \in D$. Therefore, (3.2.24) holds.

Finally, we show that v is a weak solution to problem (1.1.6). By (3.2.25), we get

$$
\begin{aligned}
\sup_{k\in\mathbf{N}} \|u_k\|_{L^2}^2 &= \sup_{k\in\mathbf{N}} \int_D \left(E_x^Q\left[W_k f(X_{\tau_D})\right]\right)^2 dx\\
&< \frac{\|f\|_\infty^2 |D|}{1-2\theta}, \qquad (3.2.26)
\end{aligned}
$$

where $|D|$ is the Lebesgue measure of D. Since u_k is the weak solution to problem (3.2.23), we have $\mathcal{E}_k(u_k, \phi) = 0$ for any $\phi \in C_0^\infty(D)$. Then, $\mathcal{E}_k(u_k, \phi) = 0, \quad \forall \phi \in H_0^{1,2}(D)$. Thus, we have $\mathcal{E}_k(u_k, u_k - u_1) = 0$, which implies that

$$
\mathcal{E}_k(u_k, u_k) = \mathcal{E}_k(u_k, u_1). \qquad (3.2.27)
$$

Note that $|b|^2$, $|\hat{b}|^2$ and c are in the Kato class. For any $0 < \varepsilon < 1$, there exists a constant $A(\varepsilon) > 1$ such that for $1 \le i \le n$ and $\eta \in H^{1,2}(\mathbf{R}^n)$ (cf. Kato (1980)),

$$
\int_{\mathbf{R}^n} (b_i^2 + \hat{b}_i^2 + |c|)\eta^2 dx \le \varepsilon \int_{\mathbf{R}^n} |\nabla \eta|^2 dx + A(\varepsilon)\int_{\mathbf{R}^n} \eta^2 dx. \qquad (3.2.28)
$$

By (3.2.28), we obtain that for $k \in \mathbf{N}$, $1 \le i \le n$ and $\eta \in H^{1,2}(\mathbf{R}^n)$,

$$
\begin{aligned}
&\int_{\mathbf{R}^n} ((\hat{b}_{k,i})^2 + |c_k|)\eta^2 dx\\
\le\ &\int_{\mathbf{R}^n} \left\{\int_{\mathbf{R}^n} [\hat{b}_i^2(x-y) + |c|(x-y)]J_k(y)dy\right\}\eta^2(x)dx\\
\le\ &\varepsilon \int_{\mathbf{R}^n} |\nabla \eta|^2 dx + A(\varepsilon)\int_{\mathbf{R}^n} \eta^2 dx. \qquad (3.2.29)
\end{aligned}
$$

Then, we obtain by (3.2.27)-(3.2.29) that for $k \in \mathbf{N}$,

$$
\frac{\lambda}{2}\|\nabla u_k\|_{L^2}^2 \le \frac{1}{2}\sum_{i,j=1}^n \int_D a_{ij}(x)\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} dx
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i,j=1}^{n}\int_D a_{ij}(x)\frac{\partial u_k}{\partial x_i}\frac{\partial u_1}{\partial x_j}dx - \sum_{i=1}^{n}\int_D b_i(x)\frac{\partial u_k}{\partial x_i}u_1(x)dx \\
&\quad -\sum_{i=1}^{n}\int_D \hat{b}_{k,i}(x)\frac{\partial u_k}{\partial x_i}u_1(x)dx - \sum_{i=1}^{n}\int_D \hat{b}_{k,i}(x)u_k(x)\frac{\partial u_1}{\partial x_i}dx \\
&\quad -\int_D c_k(x)u_k(x)u_1(x)dx + \sum_{i=1}^{n}\int_D b_i(x)\frac{\partial u_k}{\partial x_i}u_k(x)dx \\
&\quad +2\sum_{i=1}^{n}\int_D \hat{b}_{k,i}(x)\frac{\partial u_k}{\partial x_i}u_k(x)dx + \int_D c_k(x)u_k^2(x)dx \\
&\le \frac{n^2}{2\lambda}\|\nabla u_k\|_{L^2}\|\nabla u_1\|_{L^2} + 2nA^{1/2}(\varepsilon)\|\nabla u_k\|_{L^2}\|u_1\|_{H^{1,2}} \\
&\quad +nA^{1/2}(\varepsilon)\|\nabla u_1\|_{L^2}\|u_k\|_{H^{1,2}} + A(\varepsilon)\|u_k\|_{H^{1,2}}\|u_1\|_{H^{1,2}} \\
&\quad +3n\|\nabla u_k\|_{L^2}(\varepsilon\|\nabla u_k\|_{L^2}^2 + A(\varepsilon)\|u_k\|_{L^2}^2)^{1/2} \\
&\quad +(\varepsilon\|\nabla u_k\|_{L^2}^2 + A(\varepsilon)\|u_k\|_{L^2}^2). \qquad (3.2.30)
\end{aligned}
$$

Let ε be much smaller than λ. Then, we obtain by (3.2.26) and (3.2.30) that $\sup_{k\in\mathbf{N}}\|\nabla u_k\|_{L^2} < \infty$ and thus

$$
\sup_{k\in\mathbf{N}}\|u_k\|_{H^{1,2}} < \infty.
$$

By taking a subsequence if necessary, we may assume that $u_k \to v_1$ weakly in $H^{1,2}(D)$ as $k\to\infty$ and that its Cesaro mean $\{u'_k := \frac{1}{k}\sum_{l=1}^{k}u_l,\ k\ge 1\} \to v_2$ in $H^{1,2}(D)$ as $k\to\infty$. By (3.2.24) and Proposition III. 3.5 of Ma and Röckner (1992), we obtain that $v_1 = v_2 = v$ for q.e. $x\in D$ and

$$
v \text{ is quasi continuous in } D. \qquad (3.2.31)
$$

Let $\phi\in C_0^\infty(D)$. Note that for $l\in\mathbf{N}$,

$$
\begin{aligned}
\mathcal{E}_l(u_l,\phi) &= \frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n} a_{ij}(x)\frac{\partial u_l}{\partial x_i}\frac{\partial \phi}{\partial x_j}dx - \sum_{i=1}^{n}\int_{\mathbf{R}^n} b_i(x)\frac{\partial u_l}{\partial x_i}\phi(x)dx \\
&\quad -\sum_{i=1}^{n}\int_{\mathbf{R}^n}\hat{b}_{l,i}(x)\frac{\partial (u_l\phi)}{\partial x_i}dx - \int_{\mathbf{R}^n} c_l(x)u_l(x)\phi(x)dx. \qquad (3.2.32)
\end{aligned}
$$

By (3.2.28) and (3.2.29), we find that (cf. Lemma 2.2(iv) of Chen and Zhang (2009))

$$
\lim_{k\to\infty}\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n} a_{ij}(x)\frac{\partial u'_k}{\partial x_i}\frac{\partial \phi}{\partial x_j}dx = \frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n} a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial \phi}{\partial x_j}dx, \qquad (3.2.33)
$$

$$\lim_{k\to\infty}\sum_{i=1}^{n}\int_{\mathbf{R}^n} b_i(x)\frac{\partial u_k'}{\partial x_i}\phi(x)dx = \sum_{i=1}^{n}\int_{\mathbf{R}^n} b_i(x)\frac{\partial v}{\partial x_i}\phi(x)dx, \tag{3.2.34}$$

$$\lim_{k\to\infty}\frac{1}{k}\sum_{l=1}^{k}\sum_{i=1}^{n}\int_{\mathbf{R}^n} \hat{b}_{l,i}(x)\frac{\partial(u_l\phi)}{\partial x_i}dx = \sum_{i=1}^{n}\int_{\mathbf{R}^n} \hat{b}_i(x)\frac{\partial(v\phi)}{\partial x_i}dx, \tag{3.2.35}$$

and

$$\lim_{k\to\infty}\frac{1}{k}\sum_{l=1}^{k}\int_{\mathbf{R}^n} c_l(x)u_l(x)\phi(x)dx = \int_{\mathbf{R}^n} c(x)v(x)\phi(x)dx. \tag{3.2.36}$$

Therefore, we obtain by (3.1.3) and (3.2.32)-(3.2.36) that $\mathcal{E}(v,\phi) = \lim_{k\to\infty}\frac{1}{k}\sum_{l=1}^{k}\mathcal{E}_l(u_l,\phi) = 0$. □

3.2.2 Proof of the continuity of weak solution

It is well-known that any weak solution to $Lu = 0$ in D has a locally Hölder continuous version (see Morrey (1959), cf. also Morrey (1966)). Denote by v the right-hand side of (3.1.5) and denote by v^* its continuous version in D. We will show below that

$$\lim_{x\to y, x\in D} v^*(x) = f(y),\ \forall y\in\partial D. \tag{3.2.37}$$

First, we prove an important lemma based on the Dirichlet heat kernel estimates obtained by Aronson.

Suppose $n\geq 2$. Let $p_1 > n$ and $q_1 > 1$ satisfy $\frac{1}{p_1}+\frac{1}{q_1} = 1$. Then $q_1 = \frac{p_1}{p_1-1} < \frac{n}{n-1}$. We choose $0<\alpha<1$ such that

$$q_1 < \frac{n}{n-\alpha}. \tag{3.2.38}$$

Let M_1 be a constant satisfying

$$e^{|x|} \geq M_1|x|^{(n-\alpha+1)/2},\quad \forall x\in\mathbf{R}^n. \tag{3.2.39}$$

Let $p_2 > n/2$ and $q_2 > 1$ satisfy $\frac{1}{p_2}+\frac{1}{q_2} = 1$. Then $q_2 = \frac{p_2}{p_2-1} < \frac{n}{n-2}$. We choose β satisfying

$$\frac{n}{2} - 1 < \beta < \frac{n}{2q_2}. \tag{3.2.40}$$

Let M_2 be a constant satisfying

$$e^{|x|} \geq M_2|x|^{\beta},\quad \forall x\in\mathbf{R}^n, \tag{3.2.41}$$

and let M_3 be a constant satisfying

$$e^{|x|} \geq M_3|x|^{5/8}, \quad \forall x \in \mathbf{R}^n.$$

We denote by ς the diameter of D as above. By (3.2.38) and (3.2.40), we find that

$$\int_0^{\varsigma} r^{n-q_1(n-\alpha)-1}dr < \infty \quad \text{and} \quad \int_0^{\varsigma} r^{n-2\beta q_2-1}dr < \infty.$$

Denote

$$h(t,x,y) = \frac{\sigma_1}{t^{n/2}}e^{-\frac{\sigma_2|x-y|^2}{t}}, \quad (t,x,y) \in (0,\infty) \times D \times D.$$

Then, we obtain by (3.2.13) that

$$p(t,x,y) \leq h(t,x,y), \quad (t,x,y) \in (0,\infty) \times D \times D.$$

Lemma 3.4 *Let μ be a vector field on $\mathbf{R}^n$ and ν be a function on $\mathbf{R}^n$ such that $\mu, \nu \in C^\infty(\mathbf{R}^n)$.*

(i) Suppose $n \geq 2$, $p_1 > n$ and $p_2 > n/2$. Then, for $t > 0$ and $x \in D$,

$$\begin{aligned} & \left| \int_{y \in D} h(t,x,y) \operatorname{div} \mu(y) dy \right| \\ \leq & \frac{2\sigma_1}{\sigma_2^{(n-\alpha-1)/2} M_1 t^{(1+\alpha)/2}} \left(\int_0^{\varsigma} r^{n-q_1(n-\alpha)-1} dr \right)^{1/q_1} \left(\int_{y \in D} |\mu(y)|^{p_1} dy \right)^{1/p_1} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{y \in D} h(t,x,y) \nu(y) dy \right| \\ \leq & \frac{\sigma_1}{\sigma_2^{\beta} M_2 t^{n/2-\beta}} \left(\int_0^{\varsigma} r^{n-2\beta q_2-1} dr \right)^{1/q_2} \left(\int_{y \in D} |\nu(y)|^{p_2} dy \right)^{1/p_2}. \end{aligned}$$

(ii) Suppose $n = 1$. Then, for $t > 0$ and $x \in D$,

$$\left| \int_{y \in D} h(t,x,y) \operatorname{div} \mu(y) dy \right| \leq \frac{2^{3/2}\sigma_1 \sigma_2^{3/8} \varsigma^{1/4}}{M_3 t^{7/8}} \left(\int_{y \in D} |\mu(y)|^2 dy \right)^{1/2}$$

and

$$\left| \int_{y \in D} h(t,x,y) \nu(y) dy \right| \leq \frac{\sigma_1}{t^{1/2}} \int_{y \in D} |\nu(y)| dy.$$

Proof. We only prove (i). The proof of (ii) is similar so we omit it here.

By (3.2.39), we get

$$
\begin{aligned}
&\left|\int_{y\in D} h(t,x,y)\mathrm{div}\,\mu(y)dy\right| \\
= &\left|\int_{y\in D} \langle \nabla_y h(t,x,y), \mu(y)\rangle dy\right| \\
\le &\int_{y\in D} \frac{\sigma_1}{t^{n/2}e^{\sigma_2|x-y|^2/t}}\cdot\frac{2\sigma_2|x-y|}{t}|\mu(y)|dy \\
\le &\frac{2\sigma_1\sigma_2}{t^{n/2+1}}\int_{y\in D}\frac{|x-y|}{M_1(\sigma_2|x-y|^2/t)^{(n-\alpha+1)/2}}|\mu(y)|dy \\
\le &\frac{2\sigma_1}{\sigma_2^{(n-\alpha-1)/2}M_1t^{(1+\alpha)/2}}\int_{y\in D}\frac{|\mu(y)|}{|x-y|^{n-\alpha}}dy \\
\le &\frac{2\sigma_1}{\sigma_2^{(n-\alpha-1)/2}M_1t^{(1+\alpha)/2}}\left(\int_{y\in D}\frac{1}{|x-y|^{q_1(n-\alpha)}}dy\right)^{1/q_1}\left(\int_{y\in D}|\mu(y)|^{p_1}dy\right)^{1/p_1} \\
\le &\frac{2\sigma_1}{\sigma_2^{(n-\alpha-1)/2}M_1t^{(1+\alpha)/2}}\left(\int_0^{\varsigma} r^{n-q_1(n-\alpha)-1}dr\right)^{1/q_1}\left(\int_{y\in D}|\mu(y)|^{p_1}dy\right)^{1/p_1}.
\end{aligned}
$$

By (3.2.41), we get

$$
\begin{aligned}
&\left|\int_{y\in D} h(t,x,y)\nu(y)dy\right| \\
\le &\int_{y\in D}\frac{\sigma_1}{t^{n/2}e^{\sigma_2|x-y|^2/t}}|\nu(y)|dy \\
\le &\int_{y\in D}\frac{\sigma_1}{M_2t^{n/2}(\sigma_2|x-y|^2/t)^{\beta}}|\nu(y)|dy \\
= &\frac{\sigma_1}{\sigma_2^{\beta}M_2t^{n/2-\beta}}\int_{y\in D}\frac{|\nu(y)|}{|x-y|^{2\beta}}dy \\
\le &\frac{\sigma_1}{\sigma_2^{\beta}M_2t^{n/2-\beta}}\left(\int_{y\in D}\frac{1}{|x-y|^{2\beta q_2}}dy\right)^{1/q_2}\left(\int_{y\in D}|\nu(y)|^{p_2}dy\right)^{1/p_2} \\
\le &\frac{\sigma_1}{\sigma_2^{\beta}M_2t^{n/2-\beta}}\left(\int_0^{\varsigma} r^{n-2\beta q_2-1}dr\right)^{1/q_2}\left(\int_{y\in D}|\nu(y)|^{p_2}dy\right)^{1/p_2}.
\end{aligned}
$$

□

Remark 3.5 *In Cho et al. (2012), Cho, Kim and Park established very nice sharp two-sided estimates on Dirichlet heat kernels. Under the additional assumption that D is a $C^{1,\alpha}$-domain ($0 < \alpha \le 1$) satisfying the connected line condition and each a_{ij}, $1 \le i,j \le n$, is*

Dini continuous, by Theorem 1.1 of Cho et al. (2012), for each $T > 0$, there exist positive constants c_1 and c_2 such that for $(t, x, y) \in (0, T) \times D \times D$,

$$p(t,x,y) \le \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right)\frac{c_1}{t^{n/2}}e^{-\frac{c_2|x-y|^2}{t}} \tag{3.2.42}$$

and

$$|\nabla_y p(t,x,y)| \le \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\frac{c_1}{t^{(n+1)/2}}e^{-\frac{c_2|x-y|^2}{t}}, \tag{3.2.43}$$

where $\rho(x) := dist(x, \partial D)$.

By virtue of (3.2.42) and (3.2.43), we can obtain estimates for $p(t, x, y)$ similar to those for $h(t, x, y)$ given as in Lemma 3.4. These estimates for $p(t, x, y)$ or $h(t, x, y)$ make it possible to handle the case when Meyers's L^p-estimate is not available.

Proof of the continuity of weak solution at the boundary.

By (3.2.31), we have $v^*(x) = v(x)$ for q.e. $x \in D$. Note that for $x \in D$,

$$\begin{aligned} v(x) &= E_x^Q\left[\exp\left(\int_0^{\tau_D} c(X_s)ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D} \hat{b}^H(X_s)ds\right) f(X_{\tau_D})\right] \\ &= E_x^Q[f(X_{\tau_D})] + E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1)], \end{aligned}$$

where $A_t := \int_0^t c(X_s)ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s)ds$, $t \ge 0$. By Lemma 3.2, to prove (3.2.37), it suffices to show that there exists an exceptional set $F \subset D$ such that

$$\lim_{x\to y, x\in D\setminus F} E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1)] = 0, \quad \forall y \in \partial D. \tag{3.2.44}$$

For $t > 0$ and $x \in D$, we have

$$\begin{aligned} E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1)] &= E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D \le t] \\ &\quad + E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D > t]. \end{aligned}$$

By (3.2.22), there exists an exceptional set $F_1 \subset D$ such that

$$\sup_{x\in D\setminus F_1} E_x^Q[\exp(A_{\tau_D})] = \sup_{x\in D\setminus F_1} E_x[Z_{\tau_D}] \le \frac{1}{1-\theta}.$$

Then, we obtain by the strong Markov property that for q.e. $x \in D$,

$$\begin{aligned}
&\left|E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}}-1);\tau_D>t]\right| \\
\leq\ & \|f\|_\infty \left\{Q_x(\tau_D>t)+E_x^Q[e^{A_{\tau_D}};\tau_D>t]\right\} \\
\leq\ & \|f\|_\infty \left\{Q_x(\tau_D>t)+\frac{E_x^Q[e^{A_t};\tau_D>t]}{1-\theta}\right\},\ \forall t>0. \qquad (3.2.45)
\end{aligned}$$

By Lemma 3.2, following the argument of (2.28) of Chen and Zhao (1995), we get

$$\lim_{x\to y, x\in D} Q_x(\tau_D>t)=0,\ \forall t>0, \forall y\in\partial D. \qquad (3.2.46)$$

By (3.2.17), (3.2.20), (3.2.21), (3.2.25) and Fatou's lemma, there exists an exceptional set $F_2\subset D$ such that for every $t>0$,

$$\sup_{x\in D\setminus F_2} E_x^Q[e^{2A_t};\tau_D>t]\leq \sup_{x\in D\setminus F_2}\sup_{k\in\mathbf{N}} E_x^Q\left[e^{2\int_0^{\tau_D} g_k(X_s)ds}ds\right]\leq\frac{1}{1-2\theta}. \qquad (3.2.47)$$

Thus, we obtain by (3.2.45)-(3.2.47) that there exists an exceptional set $F_3\subset D$ satisfying

$$\lim_{x\to y, x\in D\setminus F_3} E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}}-1);\tau_D>t]=0,\ \forall t>0, \forall y\in\partial D.$$

Therefore, to prove (3.2.44), it suffices to show that there exists an exceptional set $F_4\subset D$ such that

$$\lim_{t\downarrow 0}\sup_{x\in D\setminus F_4} E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}}-1);\tau_D\leq t]=0. \qquad (3.2.48)$$

By (3.2.17), (3.2.20), (3.2.21) and Fatou's lemma, there exists an exceptional set $F_4\subset D$ such that for every $t>0$,

$$\begin{aligned}
&\sup_{x\in D\setminus F_4}|E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}}-1);\tau_D\leq t]| \\
\leq\ & \|f\|_\infty \sup_{x\in D\setminus F_4}\liminf_{k\to\infty} E_x^Q\left[\left|e^{\int_0^{\tau_D}(c_k-\operatorname{div}\hat{b}_k)(X_s)ds}-1\right|;\tau_D\leq t\right] \\
\leq\ & \|f\|_\infty \left\{\sup_{x\in D}\limsup_{k\to\infty} E_x^Q\left[e^{\int_0^{\tau_D} g_k(X_s)ds}-1;\tau_D\leq t\right]\right. \\
&\left.+\sup_{x\in D}\limsup_{k\to\infty} E_x^Q\left[\left(1-e^{\int_0^{\tau_D}(c_k-\operatorname{div}\hat{b}_k-g_k)(X_s)ds}\right);\tau_D\leq t\right]\right\} \\
\leq\ & \|f\|_\infty \left\{\sup_{x\in D}\limsup_{k\to\infty} E_x^Q\left[e^{\int_0^{t\wedge\tau_D} g_k(X_s)ds}-1\right]\right. \\
&\left.+\sup_{x\in D}\limsup_{k\to\infty} E_x^Q\left[\left(1-e^{\int_0^{t\wedge\tau_D}(c_k-\operatorname{div}\hat{b}_k-g_k)(X_s)ds}\right)\right]\right\}.
\end{aligned}$$

By Lemma 3.4 and Khasminskii's inequality, we get

$$\lim_{t\downarrow 0}\sup_{x\in D}\sup_{k\in\mathbf{N}} E_x^Q\left[e^{\int_0^{t\wedge\tau_D} g_k(X_s)ds}\right]=1.$$

Hence, to prove (3.2.48), we need only show that

$$\liminf_{t\downarrow 0}\inf_{x\in D}\inf_{k\in\mathbf{N}} E_x^Q\left[e^{\int_0^{t\wedge\tau_D}(c_k-\operatorname{div}\hat{b}_k-g_k)(X_s)ds}\right]\geq 1.$$

Further, by Jensen's inequality, we need only show that

$$\lim_{t\downarrow 0}\sup_{x\in D}\sup_{k\in\mathbf{N}} E_x^Q\left[\int_0^{t\wedge\tau_D}(g_k-c_k+\operatorname{div}\hat{b}_k)(X_s)ds\right]=0.$$

By Lemma 3.4, we obtain that

$$\begin{aligned}
& \sup_{x\in D}\sup_{k\in\mathbf{N}} E_x^Q\left[\int_0^{t\wedge\tau_D}(g_k-c_k+\operatorname{div}\hat{b}_k)(X_s)ds\right] \\
= & \sup_{x\in D}\sup_{k\in\mathbf{N}}\int_0^t\int_{y\in D}p(s,x,y)(g_k-c_k+\operatorname{div}\hat{b}_k)(y)dyds \\
\leq & \sup_{x\in D}\sup_{k\in\mathbf{N}}\int_0^t\int_{y\in D}h(s,x,y)(g_k-c_k+\operatorname{div}\hat{b}_k)(y)dyds \\
\to & \ 0 \text{ as } t\downarrow 0.
\end{aligned}$$

□

3.2.3 Proof of the uniqueness of continuous weak solutions

In this subsection, we will prove that there exists a unique continuous weak solution to problem (1.1.6).

Let u_1 be a weak solution to problem (1.1.6) such that u_1 is continuous on $\overline{D}$. We have Fukushima's decomposition

$$\begin{aligned}
u_1(X_t)-u_1(X_0) &= M_t^{u_1}+N_t^{u_1} \\
&= \int_0^t\nabla u_1(X_s)dM_s+N_t^{u_1},\quad t<\tau_D. \qquad (3.2.49)
\end{aligned}$$

We claim that

$$N_t^{u_1} = -\sum_{i=1}^n\int_0^t b_i(X_s)\frac{\partial u_1}{\partial x_i}(X_s)ds-\int_0^t u_1(X_s)c(X_s)ds$$

$$
-\int_0^t u_1(X_s)dN_s^{\hat{b}^H}+\int_0^t u_1(X_s)\hat{b}^H(X_s)ds,
$$
$$
t<\tau_D,\ P_x-\text{a.s. for q.e. } x\in D, \tag{3.2.50}
$$

where the third term of (3.2.50) is a Nakao integral (we refer the readers to Definition 2.4 of Chen et al. (2012) and Definition 3.1 of Nakao (1985) for the definition).

Let $\{D_n\}$ be a sequence of increasing open subsets of $\mathbf{R}^n$ satisfying $D=\cup_{n\in\mathbf{N}}D_n$ and $\overline{D_n}\subset D_{n+1}$ for each n. We choose a sequence $\{u^{(n)}\subset H_0^{1,2}(D)\cap\mathcal{B}_b(D_n)\}$ satisfying $u_1=u^{(n)}$ on D_n for each n. To prove (3.2.50) it suffices to show that for any $n\in\mathbf{N}$,

$$
\begin{aligned}
N_t^{u^{(n)}} &= -\sum_{i=1}^n\int_0^t b_i(X_s)\frac{\partial u^{(n)}}{\partial x_i}(X_s)ds-\int_0^t u^{(n)}(X_s)c(X_s)ds\\
&\quad -\int_0^t u^{(n)}(X_s)dN_s^{\hat{b}^H}+\int_0^t u^{(n)}(X_s)\hat{b}^H(X_s)ds,\\
&\qquad t<\tau_{D_n},\ P_x-\text{a.s. for q.e. } x\in D.
\end{aligned}\tag{3.2.51}
$$

Denote by $C_t^{(n)}$ the right hand side of (3.2.51). By Theorem 5.2.7 of Oshima (1988), following the argument of the proof of Theorem 2.2 of Nakao (1985), we find that to prove (3.2.51) it suffices to show that for each n,

$$
\lim_{t\downarrow 0}\frac{1}{t}E_{\phi\cdot dx}[N_t^{u^{(n)}}]=\lim_{t\downarrow 0}\frac{1}{t}E_{\phi\cdot dx}[C_t^{(n)}],\quad \forall\phi\in H_0^{1,2}(D_n)\cap\mathcal{B}_b(D_n). \tag{3.2.52}
$$

We fix an $n\in\mathbf{N}$ and $\phi\in H_0^{1,2}(D_n)\cap\mathcal{B}_b(D_n)$. By (3.1.3), (3.1.4) and (3.2.3), we get

$$
\begin{aligned}
\mathcal{E}^0(u^{(n)},\phi) &= \mathcal{E}(u^{(n)},\phi)+\sum_{i=1}^n\int_D b_i(x)\frac{\partial u^{(n)}}{\partial x_i}\phi(x)dx\\
&\quad +\sum_{i=1}^n\int_D \hat{b}_i(x)\frac{\partial(u^{(n)}\phi)}{\partial x_i}+\int_D c(x)u^{(n)}(x)\phi(x)dx,\\
&= \sum_{i=1}^n\int_D b_i(x)\frac{\partial u^{(n)}}{\partial x_i}\phi(x)dx+\int_D c(x)u^{(n)}(x)\phi(x)dx\\
&\quad -\mathcal{E}_1^0(\hat{b}^H,u^{(n)}\phi).
\end{aligned}\tag{3.2.53}
$$

We have

$$
\begin{aligned}
\lim_{t\downarrow 0}\frac{1}{t}E_{\phi\cdot dx}[N_t^{u^{(n)}}] &= \lim_{t\downarrow 0}\frac{1}{t}E_{\phi\cdot dx}[u^{(n)}(X_t)-u^{(n)}(X_0)-M_t^{u^{(n)}}]\\
&= \lim_{t\downarrow 0}\frac{1}{t}\int_D E_x[u^{(n)}(X_t)-u^{(n)}(X_0)]\phi(x)dx
\end{aligned}
$$

$$= \quad -\mathcal{E}^0(u^{(n)}, \phi) \tag{3.2.54}$$

and

$$\begin{aligned} &\lim_{t\downarrow 0}\frac{1}{t}E_{\phi\cdot dx}\left[-\sum_{i=1}^n\int_0^t b_i(X_s)\frac{\partial u^{(n)}}{\partial x_i}(X_s)ds - \int_0^t u^{(n)}(X_s)c(X_s)ds \right.\\ &\qquad\left. + \int_0^t u^{(n)}(X_s)\hat{b}^H(X_s)ds\right] \\ = \quad & -\sum_{i=1}^n\int_D b_i(x)\frac{\partial u^{(n)}}{\partial x_i}\phi(x)dx - \int_D c(x)u^{(n)}(x)\phi(x)dx \\ & + \int_D \hat{b}^H(x)u^{(n)}(x)\phi(x)dx. \end{aligned} \tag{3.2.55}$$

By Remark 2.5 of Chen et al. (2012), we get

$$\lim_{t\downarrow 0}\frac{1}{t}E_{\phi\cdot dx}\left[-\int_0^t u^{(n)}(X_s)dN_s^{\hat{b}^H}\right] = \mathcal{E}^0(\hat{b}^H, u^{(n)}\phi). \tag{3.2.56}$$

Then, (3.2.52) holds by (3.2.53)-(3.2.56). Thus, (3.2.51) and hence (3.2.50) hold.

By (3.2.49) and (3.2.50), we obtain that

$$\begin{aligned} u_1(X_t) - u_1(X_0) \quad = \quad & \int_0^t \nabla u_1(X_s)dM_s - \sum_{i=1}^n\int_0^t b_i(X_s)\frac{\partial u_1}{\partial x_i}(X_s)ds \\ & - \int_0^t u_1(X_s)c(X_s)ds - \int_0^t u_1(X_s)dN_s^{\hat{b}^H} + \int_0^t u_1(X_s)\hat{b}^H(X_s)ds, \\ & \qquad t < \tau_D,\ P_x - \text{a.s. for q.e. } x \in D. \end{aligned} \tag{3.2.57}$$

We now prove that for $t < \tau_D$,

$$d(u_1(X_t)Z_t) = u_1(X_t)Z_t(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t\nabla u_1(X_t)dM_t, \tag{3.2.58}$$

$P_x -$ a.s. for q.e. $x \in D$, where Z_t is defined as in (3.2.19).

For $k \in \mathbf{N}$ and $t > 0$, we define

$$\begin{aligned} V_t^k \quad := \quad & \int_0^t \nabla u_1(X_s)dM_s - \sum_{i=1}^n\int_0^t b_i(X_s)\frac{\partial u_1}{\partial x_i}(X_s)ds \\ & - \int_0^t u_1(c_k - \operatorname{div}\hat{b}_k)(X_s)ds \end{aligned} \tag{3.2.59}$$

and

$$Z_t^k := \exp\left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + \int_0^t (c_k - \operatorname{div}\hat{b}_k)(X_s)ds\right).$$

Then,

$$dZ_t^k = Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k(c_k - \operatorname{div}\hat{b}_k)(X_t)dt.$$

Note that both $\{V_t^k\}$ and $\{Z_t^k\}$ are semi-martingales. Applying Ito's formula, we obtain that

$$\begin{aligned} d(V_t^k Z_t^k) &= V_t^k Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k \nabla u_1(X_t)dM_t \\ &\quad + Z_t^k(V_t^k - u_1(X_t))(c_k - \operatorname{div}\hat{b}_k)(X_t)dt. \end{aligned}$$

Further, applying Ito's formula to Z_t^k, we get

$$\begin{aligned} & d((V_t^k + u_1(X_0))Z_t^k) \\ = \; & V_t^k Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k \nabla u_1(X_t)dM_t \\ & + Z_t^k(V_t^k - u_1(X_t))(c_k - \operatorname{div}\hat{b}_k)(X_t)dt \\ & + u_1(X_0)Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + u_1(X_0)Z_t^k(c_k - \operatorname{div}\hat{b}_k)(X_t)dt \\ = \; & (V_t^k + u_1(X_0))Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k \nabla u_1(X_t)dM_t \\ & + Z_t^k(V_t^k - (u_1(X_t) - u_1(X_0)))(c_k - \operatorname{div}\hat{b}_k)(X_t)dt. \end{aligned} \tag{3.2.60}$$

By (3.2.17), (3.2.20), (3.2.57), (3.2.59) and Theorem 2.7 of Chen et al. (2012), there exists a subsequence $\{k_l\}$ such that $V_t^{k_l} \to u_1(X_t) - u_1(X_0)$, $t < \tau_D$, P_x-a.s. for q.e. $x \in D$ as $l \to \infty$. Therefore, (3.2.58) holds by (3.2.60).

By (3.2.58), we know that $\{u_1(X_{t\wedge\tau_D})Z_{t\wedge\tau_D}, t \geq 0\}$ is a P_x-local martingale for q.e. $x \in D$. We claim that $\{Z_{t\wedge\tau_D}, t \geq 0\}$ is P_x-uniformly integrable for q.e. $x \in D$. Write

$$Z_{t\wedge\tau_D} = Z_{\tau_D}1_{\{\tau_D \leq t\}} + Z_t 1_{\{\tau_D > t\}}.$$

By (3.2.22), $\{Z_{\tau_D}1_{\{\tau_D \leq t\}}, t \geq 0\}$ is P_x-uniformly integrable for q.e. $x \in D$. We now show that $\{Z_t 1_{\{\tau_D > t\}}, t \geq 0\}$ is P_x-uniformly integrable for q.e. $x \in D$. Note that for q.e. $x \in D$,

$$\begin{aligned} Z_t 1_{\{\tau_D > t\}} &\leq 1_{\{\tau_D > t\}} \exp\left(\int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^{\tau_D} b^*\tilde{a}^{-1}b(X_s)ds + \int_0^{\tau_D} g(X_s)ds\right) \\ &:= 1_{\{\tau_D > t\}} Z_{\tau_D}^g. \end{aligned}$$

Hence it suffices to show that $\{1_{\{\tau_D > t\}} Z_{\tau_D}^g, t \geq 0\}$ is P_x-uniformly integrable for $x \in D$.

By the strong Markov property, we get

$$\begin{aligned}
1_{\{\tau_D>t\}}E_x[Z^g_{\tau_D}|\mathcal{F}_t] &= 1_{\{\tau_D>t\}}Z^g_t E_{X_t}[Z^g_{\tau_D}] \\
&\geq 1_{\{\tau_D>t\}}Z^g_t \inf_{x\in D} E_x[Z^g_{\tau_D}] \\
&= 1_{\{\tau_D>t\}}Z^g_t \inf_{x\in D} E^Q_x\left[\exp\left(\int_0^{\tau_D} g(X_s)ds\right)\right] \\
&\geq 1_{\{\tau_D>t\}}Z^g_t. \qquad (3.2.61)
\end{aligned}$$

By (3.2.61) and (3.2.15), we obtain that $\{1_{\{\tau_D>t\}}Z^g_{\tau_D}, t\geq 0\}$ is P_x-uniformly integrable for $x\in D$. Therefore $\{Z_{t\wedge\tau_D}, t\geq 0\}$ is P_x-uniformly integrable for q.e. $x\in D$. Since u_1 is bounded continuous, we find that $\{u_1(X_{t\wedge\tau_D})Z_{t\wedge\tau_D}, t\geq 0\}$ is a P_x-martingale for q.e. $x\in D$. Thus,

$$u_1(x) = E_x[u_1(X_{t\wedge\tau_D})Z_{t\wedge\tau_D}], \quad \text{for q.e. } x\in D.$$

Letting $t\to\infty$, we obtain that

$$u_1(x) = E_x[f(X_{\tau_D})Z_{\tau_D}], \quad \text{for q.e. } x\in D,$$

which proves the uniqueness. □

3.3 Probabilistic representation of non-symmetric semigroup

Throughout this section, we let D be an open subset of $\mathbf{R}^n$, which need not be bounded. Let L and $(\mathcal{E}, D(\mathcal{E}))$ be defined as in (3.0.1) and (3.1.3), respectively. Since $|b|^2$, $|\hat{b}|^2$ and c are in the Kato class, there exists a constant $\gamma>0$ such that $(\mathcal{E}_\gamma, D(\mathcal{E}))$ is a coercive closed form on $L^2(D;dx)$ (cf. Lunt et al. (1998)). Hence there exits a (unique) strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on $L^2(D;dx)$ which is associated with $(\mathcal{E}, D(\mathcal{E}))$. Denote by $(\mathcal{L}, D(\mathcal{L}))$ the generator of $\{T_t\}_{t\geq 0}$. Clearly $\mathcal{L}$ is formally given by L. Denote by $\{\hat{T}_t\}_{t\geq 0}$ the dual semigroup of $\{T_t\}_{t\geq 0}$ on $L^2(D;dx)$.

We define the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ as in (3.1.4). Let $X=((X_t)_{t\geq 0},(P_x)_{x\in\mathbf{R}^n})$ and $\hat{X}=(X_t)_{t\geq 0},(\hat{P}_x)_{x\in\mathbf{R}^n})$ be the Markov process and dual Markov process associated with the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ given by (3.1.4), respectively. Let M_t, $(\tilde{a}_{ij})_{i,j=1}^n$, v^H, etc. be defined the same as in Section 3.1. Denote by m the Lebesgue measure dx on $\mathbf{R}^n$. Now we can state the main result of this section.

Theorem 3.6 *For any $f, g \in L^2(D; dx)$, we have*

$$\begin{aligned}
&\int_D f(x)T_t g(x)dx \\
= \;& E_m\left[f(X_0)g(X_t)\exp\left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds\right.\right. \\
&\left.\left. + \int_0^t c(X_s)ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s)ds\right); t < \tau_D\right]. \qquad (3.3.1)
\end{aligned}$$

Proof. By (3.2.1), similar to Theorem 2.1 of Lunt et al. (1998), we can prove the following lemma on integrability of functionals of Dirichlet processes.

Lemma 3.7 *Suppose $f \in L^{r\vee 1}(D; dx)$ for some $r > n/2$ and $T > 0$. Then, there exists a constant $\varrho_1 > 0$ depending on f, r and T such that for any $0 \le t \le T$,*

$$\sup_{x\in D} E_x\left[\exp\left(\int_0^t f(X_s)ds\right); t < \tau_D\right] \le \varrho_1 e^{\varrho_1 t},$$

and

$$\sup_{x\in D} \hat{E}_x\left[\exp\left(\int_0^t f(X_s)ds\right); t < \tau_D\right] \le \varrho_1 e^{\varrho_1 t}.$$

We divide the proof of Theorem 3.6 into three cases.

Case 1: $\hat{b} = 0$.

For $g \in \mathcal{B}_b(D)$, we define

$$\begin{aligned}
P_t g(x) := E_x\Bigg[\exp\Bigg(&\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s \\
&-\frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + \int_0^t c(X_s)ds\Bigg) g(X_t); t < \tau_D\Bigg].
\end{aligned}$$

Clearly $\{P_t\}_{t\ge 0}$ is a well-defined semigroup. We now show that $\{P_t\}_{t\ge 0}$ extends to a strongly continuous semigroup on $L^2(D; dx)$, which will be also denoted by $\{P_t\}_{t\ge 0}$.

In fact, for any $g \in L^2(D; dx)$, we obtain by Lemma 3.7 that

$$\int_D (P_t g(x))^2 dx$$

$$
\begin{aligned}
&= \int_D \left(E_x \left[\exp\left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \int_0^t b^*\tilde{a}^{-1}b(X_s)ds \right) \right.\right. \\
&\qquad \left.\left. \cdot \exp\left(\frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + \int_0^t c(X_s)ds \right) g(X_t); t < \tau_D \right] \right)^2 dx \\
&\leq \int_D E_x \left[\exp\left(2\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - 2\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \right) \right] \\
&\qquad \cdot E_x \left[\exp\left(\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + 2\int_0^t c(X_s)ds \right) g^2(X_t); t < \tau_D \right] dx \\
&= \int_D g^2(x)\hat{E}_x \left[\exp\left(\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + 2\int_0^t c(X_s)ds \right); t < \tau_D \right] dx \\
&\leq \varrho_2 e^{\varrho_2 t} \int_D g^2(x)dx, \qquad (3.3.2)
\end{aligned}
$$

where $\varrho_2 > 0$ is a constant independent of g. This gives the existence of the extension of P_t to $L^2(D; dx)$. Since $C_b(D)$ is dense in $L^2(D; dx)$ and for $g \in C_b(D)$, $P_t g(x) \to g(x)$ as $t \to 0$, the continuity property of P_t follows from (3.3.2).

Define

$$
S_t = \exp\left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + \int_0^t c(X_s)ds \right)
$$

and

$$
\bar{M}_t = \int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s.
$$

Then $S_t = 1 + \int_0^t S_s d\bar{M}_s + \int_0^t S_s c(X_s)ds$. By Ito's formula, we obtain that for $u \in D(\mathcal{L})$ and $t < \tau_D$,

$$
u(X_t)S_t = u(X_0) + \int_0^t S_s dM_s^u + \int_0^t u(X_s)S_s d\bar{M}_s + \int_0^t S_s \mathcal{L}u(X_s)ds.
$$

Following the argument of the proof of Theorem 3.2 of Lunt et al. (1998), we can show that $\{P_t\}_{t\geq 0}$ coincides with $\{T_t\}_{t\geq 0}$ for this case.

Case 2: $\hat{b} \in C_0^\infty(D)$.

Similar to the proof of Theorem 3.3 of Lunt et al. (1998), we can show that for $g \in L^2(D; dx)$,

$$
\begin{aligned}
T_t g(x) &= E_x \left[\exp\left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \right.\right. \\
&\qquad \left.\left. + \int_0^t c(X_s)ds - \int_0^t \operatorname{div}\hat{b}(X_s)ds \right) g(X_t); t < \tau_D \right].
\end{aligned}
$$

The proof of this case is complete by (3.2.4).

Case 3: $|\hat{b}|^2 \in L^{p\vee 1}(D; dx)$.

By Lemma 3.3(ii), we may choose a sequence $\{\hat{b}_n \in C_0^\infty(\mathbf{R}^n)\}$ such that $|\hat{b}_n - \hat{b}|^2 \to 0$ in $L^{p\vee 1}(\mathbf{R}^n; dx)$ and $\hat{b}_n^H \to \hat{b}^H$ in $H^{1,2}(\mathbf{R}^n)$ as $n \to \infty$.

Let $\{T_t^n\}_{t\geq 0}$ be the semigroup corresponding to the quadratic form $\mathcal{E}$ with $\hat{b}_n$ in place of $\hat{b}$. Then, for $f, g \in L^2(D; dx)$, we have

$$\begin{aligned} & \int_D f(x) T_t^n g(x) dx \\ = \;& E_m\Bigg[f(X_0)g(X_t)\exp\Bigg(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \\ & + \int_0^t c(X_s)ds + N_t^{\hat{b}_n^H} - \int_0^t \hat{b}_n^H(X_s)ds\Bigg); t < \tau_D\Bigg]. \end{aligned} \tag{3.3.3}$$

By Theorem 1.3 of Röckner and Zhang (1997), the left-hand side of (3.3.3) converges to $\int_D f(x)T_t g(x)dx$ as $n \to \infty$.

We will prove below that the right-hand side of (3.3.3) converges to the right-hand side of (3.3.1) as $n \to \infty$. Define for $t \geq 0$,

$$\begin{aligned} Y_t^n \;=\;& g(X_t)\exp\Bigg(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \\ & + \int_0^t c(X_s)ds + N_t^{\hat{b}_n^H} - \int_0^t \hat{b}_n^H(X_s)ds\Bigg), \quad n \in \mathbf{N}, \end{aligned}$$

and

$$\begin{aligned} Y_t \;=\;& g(X_t)\exp\Bigg(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \\ & + \int_0^t c(X_s)ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s)ds\Bigg). \end{aligned}$$

Then, the right-hand sides of (3.3.3) and (3.3.1) equal $E_{f\cdot m}[Y_t^n; t < \tau_D]$ and $E_{f\cdot m}[Y_t; t < \tau_D]$, respectively. To complete the proof, we need only show that $\{Y_t^n 1_{t<\tau_D}\}$ is $P_{f\cdot m}$-uniformly integrable. We will establish this below by proving that $\sup_{n\in\mathbf{N}} E_{f\cdot m}[(Y_t^n)^2; t < \tau_D] < \infty$.

In fact, we obtain by Cauchy-Schwarz inequality that

$$E_{f\cdot m}[(Y_t^n)^2; t < \tau_D]$$

$$
\begin{aligned}
&= E_{f\cdot m}\Bigg[g^2(X_t)\exp\Bigg(2\int_0^t(\tilde{a}^{-1}b)^*(X_s)dM_s - \int_0^t b^*\tilde{a}^{-1}b(X_s)ds \\
&\qquad +2\int_0^t c(X_s)ds + 2N_t^{\hat{b}_n^H} - 2\int_0^t \hat{b}_n^H(X_s)ds\Bigg); t<\tau_D\Bigg] \\
&= E_{f\cdot m}\Bigg[g^2(X_t)\exp\Bigg(\frac{1}{2}\int_0^t(\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{4}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \\
&\qquad +\frac{1}{2}\int_0^t c(X_s)ds + 2N_t^{\hat{b}_n^H} - 2\int_0^t \hat{b}_n^H(X_s)ds\Bigg) \\
&\qquad \cdot\exp\Bigg(\frac{3}{2}\int_0^t(\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{3}{4}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + \frac{3}{2}\int_0^t c(X_s)ds\Bigg); t<\tau_D\Bigg] \\
&\le E_{f\cdot m}\Bigg[g^4(X_t)\exp\Bigg(\int_0^t(\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds \\
&\qquad +\int_0^t c(X_s)ds + N_t^{4\hat{b}_n{}^H} - \int_0^t 4\hat{b}_n^H(X_s)ds\Bigg); t<\tau_D\Bigg]^{1/2} \\
&\quad \cdot E_{f\cdot m}\Bigg[\exp\Bigg(3\int_0^t(\tilde{a}^{-1}b)^*(X_s)dM_s \\
&\qquad -\frac{3}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + 3\int_0^t c(X_s)ds\Bigg); t<\tau_D\Bigg]^{1/2} \\
&= \left(\int_D f(x)T_t^{n'}g^4(x)dx\right)^{1/2}\cdot E_{f\cdot m}\Bigg[\exp\Bigg(3\int_0^t(\tilde{a}^{-1}b)^*(X_s)dM_s \\
&\qquad -\frac{3}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + 3\int_0^t c(X_s)ds\Bigg); t<\tau_D\Bigg]^{1/2},
\end{aligned}
$$

where $\{T_t^{n'}\}_{t\ge 0}$ is the semigroup corresponding to the quadratic form $\mathcal{E}$ with $4\hat{b}_n$ in place of $\hat{b}$. Thus, we obtain by Theorem 1.3 of Röckner and Zhang (1997) and Lemma 3.7 that

$$
\begin{aligned}
&\sup_{n\in\mathbf{N}} E_{f\cdot m}[(Y_t^n)^2; t<\tau_D] \\
\le\ &\sup_{n\in\mathbf{N}}\left(\int_D f(x)T_t^{n'}g^4(x)dx\right)^{1/2}\cdot E_{f\cdot m}\Bigg[\exp\Bigg(3\int_0^t(\tilde{a}^{-1}b)^*(X_s)dM_s \\
&\qquad -\frac{3}{2}\int_0^t b^*\tilde{a}^{-1}b(X_s)ds + 3\int_0^t c(X_s)ds\Bigg); t<\tau_D\Bigg]^{1/2} \\
<\ &\infty.
\end{aligned}
$$

$\square$

Chapter 4

New results on Hunt's hypothesis (H) for Lévy processes

In this chapter, we first present a comparison result on Lévy processes which shows that big jumps have no effect on the validity of (H) in some sense. Based on this result and the Kanda-Forst-Rao theorem, we give examples of subordinators satisfying (H). Next, we give a new necessary and sufficient condition for (H) and obtain an extended Kanda-Forst-Rao theorem. By virtue of this theorem, we give a new class of Lévy processes satisfying (H). Finally, we construct a type of subordinators that does not satisfy Rao's condition. To the best of our knowledge, no existing criteria can be applied to this example. It suggests that maybe new ideas and methods are needed in order to completely solve Getoor's conjecture even for the case of subordinators.

4.1 A comparison result on Lévy processes

In this section, we prove a comparison result on Lévy processes which implies that big jumps have no effect on the validity of (H) in some sense.

Let X be a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a, Q, μ). Suppose that μ_1 is a finite measure on $\mathbf{R}^n \backslash \{0\}$ such that $\mu_1 \leq \mu$. Denote $\mu_2 := \mu - \mu_1$ and let X' be a

Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a', Q, μ_2), where

$$a' := a + \int_{\{|x|<1\}} x\mu_1(dx).$$

Theorem 4.1 *Let X and X' be Lévy processes defined as above. Then*

(i) they have same semipolar sets.

(ii) they have same essentially polar sets.

(iii) if both X and X' have resolvent densities, then X satisfies (H) if and only if X' satisfies (H).

Proof. Denote by ψ and ψ' the Lévy-Khintchine exponents of X and X', respectively. Then,

$$\begin{aligned}
\psi'(z) &= i\langle a', z\rangle + \frac{1}{2}\langle z, Qz\rangle + \int_{\mathbf{R}^n} \left(1 - e^{i\langle z,x\rangle} + i\langle z, x\rangle 1_{\{|x|<1\}}\right) \mu_2(dx),\\
\psi(z) &= i\langle a, z\rangle + \frac{1}{2}\langle z, Qz\rangle + \int_{\mathbf{R}^n} \left(1 - e^{i\langle z,x\rangle} + i\langle z, x\rangle 1_{\{|x|<1\}}\right) \mu(dx)\\
&= \psi'(z) + \int_{\mathbf{R}^n} \left(1 - e^{i\langle z,x\rangle}\right) \mu_1(dx). \qquad (4.1.1)
\end{aligned}$$

(i) Suppose that Y is a compound Poisson process with Lévy measure μ_1 and is independent of X'. By (4.1.1), X has the same law as that of $X' + Y$. Let T_1 be the first jumping time of Y. Then T_1 possesses an exponential distribution and thus $P(T_1 > 0) = 1$. Hence, for any set A and any point $x \in \mathbf{R}^n$, x is a regular point of A relative to X if and only if it is a regular point of A relative to X'. Therefore X and X' have same semipolar sets.

(ii) Set $C := \mu_1(\mathbf{R}^n\backslash\{0\})$. By (4.1.1), we get

$$\mathrm{Re}\psi'(z) \leq \mathrm{Re}\psi(z) \leq \mathrm{Re}\psi'(z) + C \qquad (4.1.2)$$

and

$$|\mathrm{Im}\psi(z)| \leq |\mathrm{Im}\psi'(z)| + C, \quad |\mathrm{Im}\psi'(z)| \leq |\mathrm{Im}\psi(z)| + C. \qquad (4.1.3)$$

For $\lambda > 0$, we have

$$\mathrm{Re}\left(\frac{1}{\lambda + \psi(z)}\right) = \frac{\lambda + \mathrm{Re}\psi(z)}{(\lambda + \mathrm{Re}\psi(z))^2 + (\mathrm{Im}\psi(z))^2}, \qquad (4.1.4)$$

$$\mathrm{Re}\left(\frac{1}{\lambda+\psi'(z)}\right) = \frac{\lambda+\mathrm{Re}\psi'(z)}{(\lambda+\mathrm{Re}\psi'(z))^2+(\mathrm{Im}\psi'(z))^2}. \tag{4.1.5}$$

By (4.1.2) and (4.1.3), we find that if $\lambda \geq \sqrt{2}C$ then

$$\begin{aligned}
\frac{\lambda+\mathrm{Re}\psi(z)}{(\lambda+\mathrm{Re}\psi(z))^2+(\mathrm{Im}\psi(z))^2} &\geq \frac{\lambda+\mathrm{Re}\psi'(z)}{(\lambda+\mathrm{Re}\psi'(z)+C)^2+(|\mathrm{Im}\psi'(z)|+C)^2}\\
&\geq \frac{\lambda+\mathrm{Re}\psi'(z)}{2[(\lambda+\mathrm{Re}\psi'(z))^2+2C^2+(\mathrm{Im}\psi'(z)^2]}\\
&\geq \frac{1}{4}\frac{\lambda+\mathrm{Re}\psi'(z)}{(\lambda+\mathrm{Re}\psi'(z))^2+(\mathrm{Im}\psi'(z))^2}. \qquad (4.1.6)
\end{aligned}$$

Similar to (4.1.6), we find that if $\lambda \geq 2C$ then

$$\begin{aligned}
\frac{\lambda+\mathrm{Re}\psi'(z)}{(\lambda+\mathrm{Re}\psi'(z))^2+(\mathrm{Im}\psi'(z))^2} &\geq \frac{\lambda+\mathrm{Re}\psi(z)-C}{(\lambda+\mathrm{Re}\psi(z))^2+(|\mathrm{Im}\psi(z)|+C)^2}\\
&\geq \frac{\frac{1}{2}\lambda+\mathrm{Re}\psi(z)}{(\lambda+\mathrm{Re}\psi(z))^2+2C^2+2(\mathrm{Im}\psi(z))^2}\\
&\geq \frac{1}{4}\frac{\lambda+\mathrm{Re}\psi(z)}{(\lambda+\mathrm{Re}\psi(z))^2+(\mathrm{Im}\psi(z))^2}. \qquad (4.1.7)
\end{aligned}$$

By (4.1.4)-(4.1.7), we obtain that if $\lambda \geq 2C$ then for any $z \in \mathbf{R}^n$,

$$\frac{1}{4}\,\mathrm{Re}\left(\frac{1}{\lambda+\psi'(z)}\right) \leq \mathrm{Re}\left(\frac{1}{\lambda+\psi(z)}\right) \leq 4\,\mathrm{Re}\left(\frac{1}{\lambda+\psi'(z)}\right). \tag{4.1.8}$$

By (4.1.8) and Theorem 3.3 of Hawkes (1979), we obtain that a set is essentially polar for X if and only if it is essentially polar for X'.

(iii) This is a direct consequence of (i), (ii) and Theorem 2.1 of Hawkes (1979). □

For $\delta > 0$, we define

$$B_\delta := \{x \in \mathbf{R}^n : 0 < |x| < \delta\}.$$

Corollary 4.2 *Let X_δ be a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent $(a_\delta, Q, \mu|_{B_\delta})$, where*

$$a_\delta := \begin{cases} a + \int_{\{\delta\leq|x|<1\}} x\mu(dx), & \text{if } 0<\delta<1,\\ a, & \text{if } \delta \geq 1. \end{cases}$$

Then, all the assertions of Theorem 4.1 hold with X' replaced by X_δ.

Remark 4.3 *If $\int_{|x|\le 1}|x|\mu(dx)<\infty$, then ψ can be expressed by*

$$\psi(z)=i\langle d,z\rangle+\frac{1}{2}\langle z,Qz\rangle+\int_{\mathbf{R}^n}\left(1-e^{i\langle z,x\rangle}\right)\mu(dx),$$

where $-d$ is called the drift of X. In this case, we call (d,Q,μ) the Lévy-Khintchine exponent of X. For $\delta>0$, we define B_δ and X_δ as above. Let X'_δ be a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent $(d,Q,\mu|_{B_\delta})$. We claim that X_δ and X'_δ have the same law and then all the assertions of Theorem 4.1 hold with X' replaced by X'_δ. In fact, we have

$$d=a+\int_{\{|x|<1\}}x\mu(dx). \tag{4.1.9}$$

If $0<\delta<1$, then

$$a_\delta+\int_{\{|x|<1\}}x\mu|_{B_\delta}(dx)=\left(a+\int_{\{\delta\le|x|<1\}}\mu(dx)\right)+\int_{\{|x|<\delta\}}x\mu(dx)=d; \tag{4.1.10}$$

if $\delta\ge 1$, then

$$a_\delta+\int_{\{|x|<1\}}x\mu|_{B_\delta}(dx)=a+\int_{\{|x|<1\}}x\mu(dx)=d. \tag{4.1.11}$$

By (4.1.9)-(4.1.11), we know that X_δ and X'_δ have the same Lévy-Khintchine exponent $(d,Q,\mu|_{B_\delta})$ and thus have the same law.

4.2 Examples of subordinators satisfying (H)

In this section, we will present new examples of subordinators satisfying (H) by virtue of the comparison result given in Section 4.1 and the Kanda-Forst-Rao theorem. To the best of our knowledge, which subordinators satisfy (H) is unknown in general.

Let X be a subordinator. Then, its Lévy-Khintchine exponent ψ can be expressed by

$$\psi(z)=-idz+\int_{(0,\infty)}\left(1-e^{izx}\right)\mu(dx),\ z\in\mathbf{R},$$

where $d\ge 0$ (called the drift coefficient) and μ satisfies $\int_{(0,\infty)}(1\wedge x)\mu(dx)<\infty$.

By Proposition 1.12, when we consider (H) for subordinators, we may concentrate on the case that $d=0$. Hereafter we use $c_1,c_2,\ldots$ to denote constants whose values can change from one appearance to another.

4.2.1 Special subordinators

Let X be a subordinator. Recall that the potential measure U of X is defined by

$$U(A) = E\left[\int_0^\infty 1_{\{X_t \in A\}} dt\right],\ A \subset [0, \infty).$$

For $\alpha > 0$, the α-potential measure U^α of X is defined by

$$U^\alpha(A) = E\left[\int_0^\infty e^{-\alpha t} 1_{\{X_t \in A\}} dt\right],\ A \subset [0, \infty).$$

X is called a *special subordinator* if $U|_{(0,\infty)}$ has a decreasing density with respect to the Lebesgue measure.

Theorem 4.4 *Let X be a special subordinator. Then X satisfies (H) if and only if $d = 0$.*

Proof. By Proposition 1.12, we need only prove the sufficiency. Suppose that $d = 0$. If μ is a finite measure, then X is a compound Poisson process and thus satisfies (H).

Now we consider the case that μ is an infinite measure. By Theorem 8 of Bretagnolle (1971), X does not hit points, i.e., any single point set $\{x\}$ is a polar set of X, which together with the assumption that $U|_{(0,\infty)}$ has a decreasing density with respect to the Lebesgue measure, implies that $U|_{[0,\infty)}$ has a density with respect to the Lebesgue measure. Since for any $\alpha > 0$, $U^\alpha(\cdot) \le U(\cdot)$, we obtain that for any $\alpha \ge 0$, U^α is absolutely continuous with respect to the Lebesgue measure. Then by theorem 2.1 of Hawkes (1979), we know that for any $\alpha \ge 0$, all α-excessive functions are lower semicontinuous. Therefore, by the fact that X does not hit points and Proposition (5.1), Theorem (5.3) of Blumenthal and Getoor (1970), following the same argument for stable subordinators Blumenthal and Getoor (1970), we obtain that X satisfies (H). □

4.2.2 Locally quasi-stable subordinators

Let S be a stable subordinator of index α, $0 < \alpha < 1$. Then, its Lévy-Khintchine exponent ψ_S has the form

$$\psi_S(z) = c|z|^\alpha(1 - i\,\mathrm{sgn}(z)\tan(\pi\alpha/2)),\ z \in (-\infty, \infty),$$

where $c > 0$. Its Lévy measure μ_S is absolutely continuous with respect to the Lebesgue measure dx and can be expressed by

$$\mu_S(dx) = \begin{cases} c^+ x^{-\alpha-1}dx, & \text{if } x > 0, \\ 0, & \text{if } x \le 0, \end{cases} \tag{4.2.1}$$

where $c^+ > 0$.

Definition 4.5 *Let X be a subordinator with drift 0 and Lévy measure μ. We call X a locally quasi-stable subordinator if there exist a stable subordinator S with Lévy measure μ_S, positive constants c_1, c_2, δ, and finite measures μ_1 and μ_2 on $(0, \delta)$ such that*

$$c_1\mu_S - \mu_1 \le \mu \le c_2\mu_S + \mu_2 \quad \text{on } (0, \delta).$$

Proposition 4.6 *Any locally quasi-stable subordinator satisfies (H).*

Proof. Let X, S, μ_1, μ_2 and δ be as in Definition 4.5. By Theorem 4.1 and Remark 4.3, we assume without loss of generality that $\mu|_{[\delta,\infty)} = 0$ and $\mu_1 = 0$. Denote by ψ and ψ_S the Lévy-Khintchine exponents of X and S, respectively. Let μ_S be as in (4.2.1). Then

$$\begin{aligned} \mathrm{Re}\psi(z) &= \int_0^\infty (1-\cos(zx))\mu(dx) \\ &\ge c_1\int_0^\delta (1-\cos(zx))\mu_S(dx) \\ &= c_1\left(\int_0^\infty (1-\cos(zx))\mu_S(dx) - \int_\delta^\infty (1-\cos(zx))\mu_S(dx)\right) \\ &= c_1\mathrm{Re}\psi_S(z) - K_1 \\ &= c'|z|^\alpha - K_1, \end{aligned} \tag{4.2.2}$$

where c_1, c', K_1 are positive constants.

$$\begin{aligned} |\mathrm{Im}\psi(z)| &\le \int_0^\infty |\sin(zx)|\mu(dx) \\ &\le c_2\int_0^\delta |\sin(zx)|\mu_S(dx) \\ &= c_2\int_0^\infty |\sin(zx)|\mu_S(dx) - c_2\int_\delta^\infty |\sin(zx)|\mu_S(dx) \end{aligned}$$

$$
\begin{aligned}
&\leq c_2 c \left\{ \int_0^{1/|z|} |\sin(zx)| x^{-1-\alpha} dx + \int_{1/|z|}^{\infty} |\sin(zx)| x^{-1-\alpha} dx \right\} + K_2 \\
&\leq c_2 c \left\{ |z| \int_0^{1/|z|} x^{-\alpha} dx + \int_{1/|z|}^{\infty} x^{-1-\alpha} dx \right\} + K_2 \\
&= c''|z|^{\alpha} + K_2,
\end{aligned}
\tag{4.2.3}
$$

where c_2, c'', K_2 are positive constants. By (4.2.2) and (4.2.3) we know that the Kanda-Forst condition holds for ψ. By (4.2.2) and Hartman and Wintner (1942), we know that X has bounded continuous transition densities. Therefore, X satisfies (H) by the Kanda-Forst theorem. □

Corollary 4.7 *Let φ be a Lévy-Khintchine exponent and μ be a Lévy measure of some special subordinator with drift 0 or some locally quasi-stable subordinator. Then, the Lévy process with Lévy-Khintchine exponent*

$$
\Phi(z) := \int_{(0,\infty)} \left(1 - e^{-\varphi(z)x}\right) \mu(dx) \tag{4.2.4}
$$

satisfies (H).

Proof. Let X be a Lévy process with Lévy-Khintchine exponent φ and $(T_t)_{t\geq 0}$ be a subordinator with drift 0 and Lévy measure μ, which is independent of X. Then $Y_t := X_{T_t}$ has the Lévy-Khintchine exponent Φ defined by (4.2.4). Therefore, by Theorem 1.10, Theorem 4.4 and Proposition 4.6, we obtain that Y satisfies (H). □

4.2.3 Further examples

In this subsection, we give further examples of subordinators satisfying (H) by virtue of the comparison result and Theorem 1.11.

Let $0 < \alpha < 1$ and $0 < \delta < 1$. We define

$$
\mu_T(dx) := \frac{1}{-\log(x) x^{1+\alpha}} dx, \quad 0 < x < \delta
$$

and

$$
\mu_V(dx) = \frac{-\log(x)}{x^{1+\alpha}} dx, \quad 0 < x < \delta.
$$

Let X be a subordinator with drift 0 and Lévy measure μ.

(i) If $c_1\mu_T - \mu_1 \le \mu \le c_2\mu_S + \mu_2$ on $(0,\delta)$ for some positive constants c_1, c_2 and finite measures μ_1, μ_2 on $(0,\delta)$, then X satisfies (H).

In fact, by Theorem 4.1 and Remark 4.3, we may assume without loss of generality that $\mu|_{[\delta,\infty)} = 0$ and $\mu_1 = 0$. For any $z \in \mathbf{R}$ with $|z| > 1$, we have

$$\begin{aligned}
\mathrm{Re}\psi(z) &= \int_0^\infty (1-\cos(zx))\mu(dx) \\
&\ge c_1 \int_0^\delta (1-\cos(zx))\mu_T(dx) \\
&= c_1 \int_0^\infty (1-\cos(zx))\mu_T(dx) - c_1 \int_\delta^\infty (1-\cos(zx))\mu_T(dx) \\
&\ge c_1 \int_{1/2|z|}^{1/|z|} (1-\cos(zx)) \frac{1}{-\log(x)x^{1+\alpha}} dx - K_3 \\
&\ge c_1' z^2 \int_{1/2|z|}^{1/|z|} \frac{x^2}{-\log(x)x^{1+\alpha}} dx - K_3 \\
&\ge c_1' \frac{z^2}{\log(2|z|)} \int_{1/2|z|}^{1/|z|} \frac{x^2}{x^{1+\alpha}} dx - K_3 \\
&= c_1'' \frac{|z|^\alpha}{\log(2|z|)} - K_3, \qquad (4.2.5)
\end{aligned}$$

where c_1', c_2'', K_3 are positive constants. By (4.2.3) and (4.2.5), we obtain that $|\mathrm{Im}\psi(z)| \le c^*(1+\mathrm{Re}\psi(z))\log(1+\mathrm{Re}\psi(z))$ for some positive constant c^*. By Hartman and Wintner (1942) and (4.2.5), we know that X has bounded continuous transition densities. Therefore, X satisfies (H) by Theorem 1.11.

(ii) If $c_1\mu_S - \mu_1 \le \mu \le c_2\mu_V + \mu_2$ on $(0,\delta)$ for some positive constants c_1, c_2 and finite measures μ_1, μ_2 on $(0,\delta)$, then X satisfies (H).

In fact, by Theorem 4.1 and Remark 4.3, we may assume without loss of generality that $\mu|_{[\delta,\infty)} = 0$ and $\mu_1 = 0$. For any $z \in \mathbf{R}$ with $|z| > 1/\delta$, we have

$$\begin{aligned}
|\mathrm{Im}\psi(z)| &\le \int_0^\infty |\sin(zx)|\mu(dx) \\
&\le c_2 \int_0^\delta |\sin(zx)|\mu_V(dx) + K_4 \\
&\le c_2 \left\{ \int_0^{1/|z|} |\sin(zx)| \frac{-\log(x)}{x^{1+\alpha}} dx + \int_{1/|z|}^\delta |\sin(zx)| \frac{-\log(x)}{x^{1+\alpha}} dx \right\} + K_4
\end{aligned}$$

$$
\begin{aligned}
&\leq c_2' \left\{ |z| \int_0^{1/|z|} \frac{-\log(x)}{x^\alpha} dx + \log(|z|) \int_{1/|z|}^\infty x^{-1-\alpha} dx \right\} + K_4 \\
&\leq c_2'' \left\{ |z| \int_0^{1/|z|} \frac{-(1-\alpha)\log(x) - 1}{x^\alpha} dx + |z|^\alpha \log(|z|) \right\} + K_4 \\
&= 2c_2''|z|^\alpha \log(|z|) + K_4, \qquad (4.2.6)
\end{aligned}
$$

where c_2', c_2'', K_4 are positive constants. By (4.2.2) and (4.2.6), we obtain that $|\mathrm{Im}\psi(z)| \leq c^{**}\mathrm{Re}\psi(z)\log(\mathrm{Re}\psi(z))$ for some positive constant c^{**}. By (4.2.2) and Hartman and Wintner (1942), we know that X has bounded continuous transition densities. Therefore, X satisfies (H) by Theorem 1.11.

4.3 A new necessary and sufficient condition for (H) and an extended Kanda-Forst-Rao theorem

Let X be a Lévy process on $\mathbf{R}^n$. From now on we assume that all 1-excessive functions are lower semicontinuous, equivalently, X has resolvent densities. Define

$$A := 1 + \mathrm{Re}(\psi), \quad B := |1+\psi|.$$

Theorem 4.8 (Rao (1988)) *Let ν be a finite measure of finite 1-energy, i.e.,*

$$\int_{\mathbf{R}^n} B^{-2}(z)A(z)|\hat{\nu}(z)|^2 dz < \infty.$$

Then

$$\lim_{\lambda\to\infty} \int_{\mathbf{R}^n} |\hat{\nu}(z)|^2(\lambda + \mathrm{Re}\psi(z))|\lambda+\psi(z)|^{-2} dz \qquad (4.3.1)$$

exists. The limit is zero if and only if $U^1\nu$ is regular.

Based on Theorems 4.8 and 1.11, we can prove the following result.

Lemma 4.9 *Let ν be a finite measure of finite 1-energy and f be an increasing function on $[1,\infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for some $N \geq 1$. Then $U^1\nu$ is regular if and only if*

$$\lim_{\lambda\to\infty} \sum_{k=1}^\infty \int_{\{B(z)>A(z)f(A(z)),\, k\leq \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, A(z)\leq\lambda<(k+1)|\mathrm{Im}\psi(z)|\}} \frac{\lambda}{\lambda^2 + (\mathrm{Im}\psi(z))^2} |\hat{\nu}(z)|^2 dz = 0.$$

Proof. Since f is an increasing function on $[1,\infty)$, $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for some $N \geq 1$ if and only if $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for any $N \geq 1$. From the proof of Theorem 1.11 (see Rao (1988)), we know that the limit

$$\lim_{\lambda\to\infty} \int_{A(z)\leq\lambda} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz \tag{4.3.2}$$

exists and equals the limit in (4.3.1). We now show that the limit in (4.3.2) equals 0 if and only if

$$\lim_{\lambda\to\infty} \int_{\{A(z)\leq\lambda,\, B(z)>A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz = 0. \tag{4.3.3}$$

To this end, we need only show that (4.3.3) implies that

$$\lim_{\lambda\to\infty} \int_{A(z)\leq\lambda} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz = 0. \tag{4.3.4}$$

Suppose that (4.3.3) holds. Then, the limit

$$\lim_{\lambda\to\infty} \int_{\{A(z)\leq\lambda,\, B(z)\leq A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz$$

exists since the limit in (4.3.2) always exists. Note that

$$\begin{aligned}
& \int_1^\infty \lambda^{-1} f(\lambda)^{-1} d\lambda \int_{\{A(z)\leq\lambda,\, B(z)\leq A(z)f(A(z))\}} \lambda(\lambda^2 + B^2(z))^{-1} |\hat{\nu}(z)|^2 dz \\
= \;& \int_{\{B(z)\leq A(z)f(A(z))\}} |\hat{\nu}(z)|^2 dz \int_{A(z)}^\infty [f(\lambda)(\lambda^2 + B^2(z))]^{-1} d\lambda \\
\leq \;& \frac{\pi}{2} \int_{\{B(z)\leq A(z)f(A(z))\}} [B(z)f(A(z))]^{-1} |\hat{\nu}(z)|^2 dz \\
\leq \;& \frac{\pi}{2} \int_{\mathbf{R}^d} B^{-2}(z) A(z) |\hat{\nu}(z)|^2 dz \\
< \;& \infty.
\end{aligned}$$

Since $\int_1^\infty \lambda^{-1} f(\lambda)^{-1} d\lambda = \infty$,

$$\lim_{\lambda\to\infty} \int_{\{A(z)\leq\lambda,\, B(z)\leq A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz = 0.$$

Therefore, (4.3.4) holds by (4.3.3).

For each $k \in \mathbf{N}$, we have

$$1_{\{k\leq \frac{|\mathrm{Im}\psi(z)|}{A(z)} < k+1,\, \lambda\geq(k+1)|\mathrm{Im}\psi(z)|\}} \frac{\lambda}{\lambda^2 + (\mathrm{Im}\psi(z))^2} |\hat{\nu}(z)|^2$$

$$
\begin{aligned}
&\le \quad 1_{\{k\le\frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\,\lambda\ge(k+1)|\mathrm{Im}\psi(z)|\}}\frac{1}{\lambda}|\hat{\nu}(z)|^2 \\
&\le \quad \frac{1}{k+1}1_{\{k\le\frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1\}}\frac{|\hat{\nu}(z)|^2}{|\mathrm{Im}\psi(z)|}.
\end{aligned}
\tag{4.3.5}
$$

We assume without loss of generality that $f(1)=\sqrt{2}$. Note that $B(z)>A(z)f(A(z))$ implies that $B(z)\le\sqrt{2}|\mathrm{Im}\psi(z)|$. Then, we obtain by $\int_{\mathbf{R}^n}B^{-2}(z)A(z)|\hat{\nu}(z)|^2dz<\infty$ that

$$
\sum_{k=1}^{\infty}\frac{1}{2(k+1)}\int_{\{B(z)>A(z)f(A(z)),\,k\le\frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1\}}\frac{|\hat{\nu}(z)|^2}{|\mathrm{Im}\psi(z)|}dz<\infty. \tag{4.3.6}
$$

By (4.3.5), (4.3.6) and the dominated convergence theorem, we get

$$
\lim_{\lambda\to\infty}\sum_{k=1}^{\infty}\int_{\{B(z)>A(z)f(A(z)),\,k\le\frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\,\lambda\ge(k+1)|\mathrm{Im}\psi(z)|\}}\frac{\lambda}{\lambda^2+(\mathrm{Im}\psi(z))^2}|\hat{\nu}(z)|^2dz=0.
$$

Therefore, the proof is complete by noting (4.3.3). $\square$

Note that if ν is a finite measure such that $U^1\nu$ is bounded then ν has finite 1-energy (cf. Rao (1988)). By Lemma 4.9 and Proposition 1.9, we obtain the following necessary and sufficient condition for (H).

Theorem 4.10 *Let f be an increasing function on $[1,\infty)$ such that $\int_N^\infty(\lambda f(\lambda))^{-1}d\lambda=\infty$ for some $N\ge1$. Then (H) holds if and only if*

$$
\lim_{\lambda\to\infty}\sum_{k=1}^{\infty}\int_{\{B(z)>A(z)f(A(z)),k\le\frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,A(z)\le\lambda<(k+1)|\mathrm{Im}\psi(z)|\}}\frac{\lambda}{\lambda^2+(\mathrm{Im}\psi(z))^2}|\hat{\nu}(z)|^2dz=0 \tag{4.3.7}
$$

for any finite measure ν with compact support such that $U^1\nu$ is bounded.

Remark 4.11 *Theorem 4.10 indicates that the validity of (H) is closely related to the behavior of $\psi(z)$ where $\mathrm{Im}(\psi(z))$ is not well controlled by $\mathrm{Re}(\psi(z))$, which is possible and can be seen from the uniform motion on $\mathbf{R}$ and the example given in Section 5.*

By virtue of Theorem 4.10, we obtain the following result extending the Kanda-Forst-Rao theorem on (H).

Theorem 4.12 *(H) holds if the following extended Kanda-Forst-Rao condition ((EKFR) for short) holds:*

(EKFR) There are two measurable functions ψ_1 and ψ_2 on $\mathbf{R}^n$ such that $\mathrm{Im}(\psi) = \psi_1 + \psi_2$, and

$$\begin{aligned} &|\psi_1| \le Af(A), \\ &\int_{\mathbf{R}^n} \frac{|\psi_2(z)|}{(1+\mathrm{Re}\psi(z))^2 + (\mathrm{Im}\psi(z))^2} dz < \infty, \end{aligned} \tag{4.3.8}$$

where f is an increasing function on $[1,\infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for some $N \ge 1$.

Proof. By Theorem 4.10, we need only show that the limit in (4.3.7) equals 0. We assume without loss of generality that $f(1) = 1/3$. Note that $B(z) > 3\sqrt{2}A(z)f(A(z))$ implies that $|\mathrm{Im}\psi(z)| > A(z)$ and $|\mathrm{Im}\psi(z)| > B(z)/\sqrt{2}$, and $|\psi_2(z)| > 2A(z)f(A(z))$ implies that $|\psi_2(z)| > |\mathrm{Im}\psi(z)|/2$. Then, by (4.3.8), the fact that $A(z) \le c(1+|z|^2)$ for some positive constant c and the dominated convergence theorem, we obtain that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_{\{B(z)>3\sqrt{2}A(z)f(A(z)),\, k\le \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, A(z)\le\lambda<(k+1)|\mathrm{Im}\psi(z)|\}} \frac{\lambda}{\lambda^2 + (\mathrm{Im}\psi(z))^2} |\hat{\nu}(z)|^2 dz \\
\le\; &\sum_{k=1}^{\infty} \int_{\{|\mathrm{Im}\psi(z)|>3A(z)f(A(z)),\, k\le \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, A(z)\le\lambda<(k+1)|\mathrm{Im}\psi(z)|\}} \frac{1}{2|\mathrm{Im}\psi(z)|} |\hat{\nu}(z)|^2 dz \\
\le\; &\sum_{k=1}^{\infty} \int_{\{|\psi_2(z)|>2A(z)f(A(z)),\, k\le \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, A(z)\le\lambda<(k+1)|\mathrm{Im}\psi(z)|\}} \frac{|\psi_2(z)|}{|\mathrm{Im}\psi(z)|^2} |\hat{\nu}(z)|^2 dz \\
\le\; &\sum_{k=1}^{\infty} \int_{\{k\le \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, A(z)\le\lambda<(k+1)|\mathrm{Im}\psi(z)|\}} \frac{2|\psi_2(z)|}{B^2(z)} |\hat{\nu}(z)|^2 dz \\
\le\; &\sum_{k=1}^{\infty} \int_{\{k\le \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, \lambda<(k+1)^2A(z)\}} \frac{2|\psi_2(z)|}{B^2(z)} |\hat{\nu}(z)|^2 dz \\
\le\; &\sum_{k=1}^{\infty} \int_{\{k\le \frac{|\mathrm{Im}\psi(z)|}{A(z)}<k+1,\, \lambda<c(k+1)^2(1+|z|^2)\}} \frac{2|\psi_2(z)|}{B^2(z)} |\hat{\nu}(z)|^2 dz \\
\to\; &0 \text{ as } \lambda\to\infty.
\end{aligned}$$

The proof is complete. □

Remark 4.13 *If $\psi_2 = 0$, then the (EKFR) condition is just Rao's condition. In particular, if $f = 1$, then it is just the Kanda-Forst condition. An integrability condition similar to (4.3.8) has been used in Theorem 3.1 of Glover (1981).*

In the following, we give an application of Theorem 4.12.

Theorem 4.14 *Let $\gamma > 0$ and X be a Lévy process on* $\mathbf{R}$ *satisfying*

$$\liminf_{|z|\to\infty} \frac{\mathrm{Re}\psi(z)}{|z|\log^{\gamma}(|z|)} > 0. \tag{4.3.9}$$

Then X satisfies (H).

Proof. By (4.3.9), we get

$$\lim_{|z|\to\infty} \frac{\mathrm{Re}\psi(z)}{\log(1+|z|)} = \infty.$$

Hence X has bounded continuous transition densities by Hartman and Wintner (1942). Let $f(\lambda) = 2\log(\lambda)$ for $\lambda \in [1,\infty)$ and set $\psi_1(z) := 1_{\{|\psi(z)|\le A(z)f(A(z))\}}\mathrm{Im}\psi(z)$, $\psi_2(z) := 1_{\{|\psi(z)|> A(z)f(A(z))\}}\mathrm{Im}\psi(z)$ for $z \in \mathbf{R}$. Condition (4.3.9) implies that there exists a constant $c > 0$ such that

$$|\psi_2(z)| \ge c 1_{\{|\psi(z)|> A(z)f(A(z))\}}|z|\log^{1+\gamma}(|z|)$$

when $|z|$ is sufficiently large. Therefore, (4.3.8) holds and the proof is complete by Theorem 4.12. □

Example 4.15 *By Theorem 4.14, Theorem 4.1 and Corollary 4.2, we obtain a new class of 1-dimensional Lévy processes satisfying (H). Let X be a Lévy process on* $\mathbf{R}$ *with Lévy-Khintchine exponent (a, Q, μ). Suppose that there exist constants $\gamma > 0$, $0 < \delta < 1$, $c > 0$, and a finite measure μ' on $\{x \in \mathbf{R}^n : 0 < |x| < \delta\}$ such that*

$$d\mu \ge \frac{c(-\log(|x|))^{\gamma}}{x^2}dx - d\mu' \quad \text{on } \{x \in \mathbf{R} : 0 < |x| < \delta\}.$$

Similar to (4.2.5), we can show that (4.3.9) holds. Then, X satisfies (H). Note that in this example it does not matter if a or Q equals 0.

Let Y be another 1-dimensional Lévy process which is independent of X. Theorem 4.14 implies that the perturbed process $Y + X$ also satisfies (H).

Remark 4.16 *Blumenthal and Getoor (1961) introduced the following index β'' defined by*

$$\beta'' = \sup\left\{\tau \ge 0 : \frac{\mathrm{Re}\psi(z)}{|z|^{\tau}} \to \infty \text{ as } |z| \to \infty\right\}. \tag{4.3.10}$$

Let X be a Lévy process on $\mathbf{R}$*. Then, Theorem 4.14 implies that (H) holds when* $\beta'' > 1$*. This result is also a direct consequence of the following proposition.*

Proposition 4.17 *Let X be a Lévy process on* $\mathbf{R}$*. Suppose that*

$$\liminf_{|z|\to\infty} \frac{|\psi(z)|}{|z| \log^{1+\gamma} |z|} > 0 \tag{4.3.11}$$

for some constant $\gamma > 0$. Then (H) holds.

Proof. *Set* $\psi_1(z) := 1_{\{|\mathrm{Im}\psi(z)| \le A(z) f(A(z))\}} \mathrm{Im}\psi(z)$, $\psi_2(z) := 1_{\{|\mathrm{Im}\psi(z)| > A(z) f(A(z))\}} \mathrm{Im}\psi(z)$ *for* $z \in \mathbf{R}$*. Let $f \equiv 1$, condition (4.3.11) implies that*

$$\limsup_{|z|\to\infty} \left\{ \frac{|\psi_2(z)|}{(1 + \mathrm{Re}\psi(z))^2 + (\mathrm{Im}\psi(z))^2} \cdot |z| \log^{1+\gamma} |z| \right\} < \infty.$$

Therefore, (4.3.8) holds and the proof is complete by Theorem 4.12. $\square$

We remark that Proposition 4.17 can also be proved by Theorem 4.8. In fact, the limit in (4.3.1) equals the limit in (4.3.2) and hence equals 0 by (4.3.11) and the dominated convergence theorem.

4.4 A type of subordinators that does not satisfy Rao's condition

As pointed out in Rao (1988), from the proof of Theorem 1.11 it seems that the condition $B \le Af(A)$ is not far from being necessary. In this section, however, we will construct a type of subordinators that does not satisfy Rao's condition.

4.4.1 Construction of the example

We fix an α such that $\frac{1}{2} < \alpha < 1$. In the sequel, we define a function ρ on $\mathbf{R}$ which will be used as the density function of a Lévy measure μ.

First, we set $n_1 = 2$. Define a function ρ_1 on $\mathbf{R}$ as follows.

$$\rho_1(x) = \frac{1}{x^{1+\alpha}}, \quad \text{if } \frac{1}{2n_1^2} < x < \frac{1}{n_1^2}; \quad 0, \text{ otherwise.}$$

We define $\mu_1(dx) = \rho_1(x)dx$ and denote by ψ_1 the Lévy-Khintchine exponent of μ_1. Then, for $z \in [\frac{n_1}{2}, 2n_1]$, we have

$$\begin{aligned}
\mathrm{Re}\psi_1(z) &= \int_0^1 (1-\cos(zx))\mu_1(dx) \\
&\leq \frac{1}{2}\int_{1/2n_1^2}^{1/n_1^2} z^2x^2\frac{1}{x^{1+\alpha}}dx \\
&\leq \frac{2n_1^{2\alpha-2}}{2-\alpha} \\
&\leq 2
\end{aligned} \tag{4.4.1}$$

and

$$\begin{aligned}
\mathrm{Im}\psi_1(z) &= -\int_0^1 \sin(zx)\mu_1(dx) \\
&= -\int_{1/2n_1^2}^{1/n_1^2} \sin(zx)\mu_1(dx) \\
&\leq -\int_{1/2n_1^2}^{1/n_1^2} \frac{zx}{2x^{1+\alpha}}dx \\
&\leq -\frac{1}{8}n_1^{2\alpha-1}.
\end{aligned} \tag{4.4.2}$$

We increase n_1 so that $\frac{1}{8}n_1^{2\alpha-1} > \frac{6}{1-\alpha}$.

For any $z \in \mathbf{R}$, we have

$$\mathrm{Re}\psi_1(z) = \int_0^1 (1-\cos(zx))\mu_1(dx) \leq \int_{1/2n_1^2}^1 \frac{1}{x^{1+\alpha}}dx \leq \frac{2^\alpha n_1^{2\alpha}}{\alpha} \leq 4n_1^{2\alpha} \tag{4.4.3}$$

and

$$|\mathrm{Im}\psi_1(z)| \leq \int_0^1 |\sin(zx)|\mu_1(dx) \leq \int_{1/2n_1^2}^1 \frac{1}{x^{1+\alpha}}dx \leq \frac{2^\alpha n_1^{2\alpha}}{\alpha} \leq 4n_1^{2\alpha}. \tag{4.4.4}$$

We choose an $n_2 \in \mathbf{N}$ such that $n_2^2 > 2n_1^2$. We define a function ρ_2 on $\mathbf{R}$ as follows.

$$\rho_2(x) = \frac{1}{x^{1+\alpha}}, \quad \text{if } \frac{1}{2n_2^2} < x < \frac{1}{n_2^2}; \quad 0, \text{ otherwise.}$$

Note that there is no overlap between ρ_1 and ρ_2. We define $\mu_2(dx) = \rho_2(x)dx$ and denote by ψ_2 the Lévy-Khintchine exponent of μ_2. Then, similar to the above, we can show that for $z \in [\frac{n_2}{2}, 2n_2]$

$$\mathrm{Re}\psi_2(z) \leq 2 \text{ and } \mathrm{Im}\psi_2(z) \leq -\frac{1}{8}n_2^{2\alpha-1}\left(< -\frac{6}{1-\alpha}\right). \tag{4.4.5}$$

Note that for $z \in [\frac{n_1}{2}, 2n_1]$ we have

$$
\begin{aligned}
\mathrm{Re}\psi_2(z) &= \int_0^1 (1-\cos(zx))\mu_2(dx) \\
&\leq \frac{1}{2}\int_{1/2n_2^2}^{1/n_2^2} z^2x^2 \frac{1}{x^{1+\alpha}}dx \\
&\leq \frac{2n_1^2 n_2^{2\alpha-4}}{2-\alpha}
\end{aligned}
\tag{4.4.6}
$$

and

$$
\begin{aligned}
|\mathrm{Im}\psi_2(z)| &\leq \int_0^1 |\sin(zx)|\mu_2(dx) \\
&\leq \int_{1/2n_2^2}^{1/n_2^2} |\sin(zx)| \frac{1}{x^{1+\alpha}}dx \\
&\leq \int_{1/2n_2^2}^{1/n_2^2} 2n_1 x \frac{1}{x^{1+\alpha}}dx \\
&\leq \frac{2n_1 n_2^{2\alpha-2}}{1-\alpha}.
\end{aligned}
\tag{4.4.7}
$$

We increase n_2 (with n_1 fixed) so that $n_2 \geq n_1^{5/(2-2\alpha)}$. By (4.4.6) and (4.4.7), we get

$$
\mathrm{Re}\psi_2(z) \leq \frac{2}{(1-\alpha)n_1^4}, \quad |\mathrm{Im}\psi_2(z)| \leq \frac{2}{(1-\alpha)n_1^4}, \quad z \in \left[\frac{n_1}{2}, 2n_1\right]. \tag{4.4.8}
$$

Then, by (4.4.1), (4.4.2) and (4.4.8), we obtain that for $z \in [\frac{n_1}{2}, 2n_1]$,

$$
\mathrm{Re}\psi_1(z) + \mathrm{Re}\psi_2(z) \leq 2 + \frac{2}{(1-\alpha)n_1^4} \tag{4.4.9}
$$

and

$$
\mathrm{Im}\psi_1(z) + \mathrm{Im}\psi_2(z) \leq -\frac{1}{8}n_1^{2\alpha-1} + \frac{2}{(1-\alpha)n_1^4}. \tag{4.4.10}
$$

We further increase n_2 so that $n_2 \geq (96)^{1/(2\alpha-1)} n_1^{(4+2\alpha)/(2\alpha-1)}$ which ensures that for any $z \in \mathbf{R}$ (cf. (4.4.3), (4.4.4) and (4.4.5)),

$$
\mathrm{Re}\psi_1(z) \leq -\frac{1}{3n_1^4}\mathrm{Im}\psi_2\left(\frac{n_2}{2}\right), \quad |\mathrm{Im}\psi_1(z)| \leq -\frac{1}{3n_1^4}\mathrm{Im}\psi_2\left(\frac{n_2}{2}\right). \tag{4.4.11}
$$

By (4.4.5) and (4.4.11), we obtain that for $z \in [\frac{n_2}{2}, 2n_2]$,

$$
\mathrm{Re}\psi_1(z) + \mathrm{Re}\psi_2(z) \leq -\frac{1}{3n_1^4}\mathrm{Im}\psi_2\left(\frac{n_2}{2}\right) + 2 \tag{4.4.12}
$$

and

$$\mathrm{Im}\psi_1(z) + \mathrm{Im}\psi_2(z) \leq \left(1 - \frac{1}{3n_1^4}\right) \mathrm{Im}\psi_2\left(\frac{n_2}{2}\right). \tag{4.4.13}$$

Define

$$\vartheta := \max\left\{\frac{5}{2-2\alpha}, \frac{4+2\alpha}{2\alpha-1}\right\}. \tag{4.4.14}$$

We can set n_2 to be cn_1^{ϑ}, for some positive constant c depending only on α, such that (4.4.9), (4.4.10), (4.4.12) and (4.4.13) hold.

For any $z \in \mathbf{R}$, we have

$$\mathrm{Re}\psi_2(z) = \int_0^1 (1 - \cos(zx))\mu_2(dx) \leq \int_{1/2n_2^2}^1 \frac{1}{x^{1+\alpha}} dx \leq \frac{2^\alpha n_2^{2\alpha}}{\alpha} \leq 4n_2^{2\alpha} \tag{4.4.15}$$

and

$$|\mathrm{Im}\psi_2(z)| \leq \int_0^1 |\sin(zx)|\mu_2(dx) \leq \int_{1/2n_2^2}^1 \frac{1}{x^{1+\alpha}} dx \leq \frac{2^\alpha n_2^{2\alpha}}{\alpha} \leq 4n_2^{2\alpha}. \tag{4.4.16}$$

We choose an $n_3 \in \mathbf{N}$ such that $n_2^3 > 2n_2^2$. We define a function ρ_3 on $\mathbf{R}$ as follows.

$$\rho_3(x) = \frac{1}{x^{1+\alpha}}, \quad \text{if } \frac{1}{2n_3^2} < x < \frac{1}{n_3^2}; \quad 0, \text{ otherwise.}$$

Note that there is no overlap among ρ_1, ρ_2 and ρ_3. We define $\mu_3(dx) = \rho_3(x)dx$ and denote by ψ_3 the Lévy-Khintchine exponent of μ_3. Then, similar to the above, we can show that for $z \in [\frac{n_3}{2}, 2n_3]$,

$$\mathrm{Re}\psi_3(z) \leq 2 \quad \text{and} \quad \mathrm{Im}\psi_3(z) \leq -\frac{1}{8} n_3^{2\alpha-1}, \tag{4.4.17}$$

and for any $z \in \mathbf{R}$,

$$\mathrm{Re}\psi_3(z) \leq 4n_3^{2\alpha}, \quad |\mathrm{Im}\psi_3(z)| \leq 4n_3^{2\alpha}.$$

Similar to (4.4.6) and (4.4.7), we obtain that for $z \in [\frac{n_1}{2}, 2n_1]$,

$$\mathrm{Re}\psi_3(z) \leq \frac{2n_1^2 n_3^{2\alpha-4}}{2-\alpha}, \quad |\mathrm{Im}\psi_3(z)| \leq \frac{2n_1 n_3^{2\alpha-2}}{1-\alpha} \tag{4.4.18}$$

and for $z \in [\frac{n_2}{2}, 2n_2]$,

$$\mathrm{Re}\psi_3(z) \leq \frac{2n_2^2 n_3^{2\alpha-4}}{2-\alpha}, \quad |\mathrm{Im}\psi_3(z)| \leq \frac{2n_2 n_3^{2\alpha-2}}{1-\alpha}. \tag{4.4.19}$$

We increase n_3 (with n_1, n_2 fixed) so that $n_3 \geq n_2^{5/(2-2\alpha)}$. By (4.4.18) and (4.4.19), we get

$$\mathrm{Re}\psi_3(z) \leq \frac{2}{(1-\alpha)n_2^4}, \quad |\mathrm{Im}\psi_3(z)| \leq \frac{2}{(1-\alpha)n_2^4}, \quad z \in \left[\frac{n_1}{2}, 2n_1\right] \bigcup \left[\frac{n_2}{2}, 2n_2\right]. \tag{4.4.20}$$

Hence, by (4.4.9), (4.4.10) and (4.4.20), we obtain that for $z \in [\frac{n_1}{2}, 2n_1]$,

$$\mathrm{Re}\psi_1(z) + \mathrm{Re}\psi_2(z) + \mathrm{Re}\psi_3(z) \leq 2 + \frac{2}{(1-\alpha)n_1^4} + \frac{2}{(1-\alpha)n_2^4} \tag{4.4.21}$$

and

$$\mathrm{Im}\psi_1(z) + \mathrm{Im}\psi_2(z) + \mathrm{Im}\psi_3(z) \leq -\frac{1}{8}n_1^{2\alpha-1} + \frac{2}{(1-\alpha)n_1^4} + \frac{2}{(1-\alpha)n_2^4}. \tag{4.4.22}$$

By (4.4.12), (4.4.13), (4.4.20) and (4.4.5), we obtain that for $z \in [\frac{n_2}{2}, 2n_2]$,

$$\mathrm{Re}\psi_1(z) + \mathrm{Re}\psi_2(z) + \mathrm{Re}\psi_3(z) \leq -\frac{1}{3n_1^4}\mathrm{Im}\psi_2\left(\frac{n_2}{2}\right) + 2 + \frac{2}{(1-\alpha)n_2^4} \tag{4.4.23}$$

and

$$\mathrm{Im}\psi_1(z) + \mathrm{Im}\psi_2(z) + \mathrm{Im}\psi_3(z) \leq \left(1 - \frac{1}{3n_1^4} - \frac{1}{3n_2^4}\right)\mathrm{Im}\psi_2\left(\frac{n_2}{2}\right). \tag{4.4.24}$$

We further increase n_3 so that $n_3 \geq (192)^{1/(2\alpha-1)} n_2^{(4+2\alpha)/(2\alpha-1)}$ which ensures that for any $z \in \mathbf{R}$ (cf. (4.4.3), (4.4.4), (4.4.15), (4.4.16) and (4.4.17)),

$$\mathrm{Re}\psi_1(z), \mathrm{Re}\psi_2(z), |\mathrm{Im}\psi_1(z)|, |\mathrm{Im}\psi_2(z)| \leq -\frac{1}{6n_2^4}\mathrm{Im}\psi_2\left(\frac{n_3}{2}\right). \tag{4.4.25}$$

Therefore, we obtain by (4.4.17) and (4.4.25) that for $z \in [\frac{n_3}{2}, 2n_3]$,

$$\mathrm{Re}\psi_1(z) + \mathrm{Re}\psi_2(z) + \mathrm{Re}\psi_3(z) \leq -\frac{1}{3n_2^4}\mathrm{Im}\psi_3\left(\frac{n_3}{2}\right) + 2 \tag{4.4.26}$$

and

$$\mathrm{Im}\psi_1(z) + \mathrm{Im}\psi_2(z) + \mathrm{Im}\psi_3(z) \leq \left(1 - \frac{1}{3n_1^4} - \frac{1}{3n_2^4}\right)\mathrm{Im}\psi_3\left(\frac{n_3}{2}\right). \tag{4.4.27}$$

We set n_3 to be $2^{1/(2\alpha-1)}cn_2^{\vartheta}$, where ϑ and c are as the same as above.

Continue in this way, we define $\rho_4, \rho_5, \ldots$ All of these functions have no overlap and we have estimates similar to (4.4.21)-(4.4.24), (4.4.26) and (4.4.27). Now we define

$$\rho = \sum_{i=1}^{\infty} \rho_i.$$

One finds that $\mu(dx) = \rho(x)dx$ is the Lévy measure of a subordinator X with the Lévy-Khintchine exponent

$$\psi = \sum_{i=1}^{\infty} \psi_i.$$

Moreover, we have that for $k \geq 2$,

$$n_k = (k-1)^{1/(2\alpha-1)} c n_{k-1}^{\vartheta}, \tag{4.4.28}$$

and for $z \in [\frac{n_k}{2}, 2n_k]$,

$$\mathrm{Im}\psi_k(z) \leq -\frac{1}{8} n_k^{2\alpha-1}, \tag{4.4.29}$$

$$\mathrm{Re}\psi(z) \leq -\frac{1}{3n_{k-1}^4} \mathrm{Im}\psi_k\left(\frac{n_k}{2}\right) + 2 + \frac{2}{1-\alpha} \sum_{k=1}^{\infty} \frac{1}{n_k^4}, \tag{4.4.30}$$

and

$$\mathrm{Im}\psi(z) \leq \left(1 - \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{n_k^4}\right) \mathrm{Im}\psi_k\left(\frac{n_k}{2}\right). \tag{4.4.31}$$

4.4.2 Discussions

In this subsection, we make discussion about the subordinators constructed in Subsection 4.4.1. Below we use $c_1, c_2, \ldots$ to denote positive constants depending only on α.

1. By the estimates (4.4.30) and (4.4.31), we can show that Rao's condition does not hold for the subordinators. In fact, by (4.4.28), there exists a constant $c_1 > 1$ such that

$$n_k > c_1^{c_1^k}, \quad k \in \mathbf{N}. \tag{4.4.32}$$

By (4.4.29), (4.4.30) and (4.4.31), we find that there exist constants $c_2, c_3, c_4 > 0$ such that for any $k \geq 2$,

$$\frac{-\mathrm{Im}\psi(z)}{1 + \mathrm{Re}\psi(z)} \geq c_2 n_{k-1}^4 \geq c_3 n_k^{3/\vartheta} \geq c_3 \left(\frac{z}{2}\right)^{3/\vartheta}, \quad \forall z \in [n_k/2, 2n_k]. \tag{4.4.33}$$

$$\mathrm{Re}\psi(z) \leq c_4 n_{k-1}^{\alpha\vartheta-3}, \quad \forall z \in [n_k/2, 2n_k]. \tag{4.4.34}$$

The estimates (4.4.33) and (4.4.34) imply that there does not exist an increasing function f on $[1, \infty)$ satisfying $\int_N^{\infty} (\lambda f(\lambda))^{-1} d\lambda = \infty$ for some $N \geq 1$ and $|1+\psi| \leq (1+\mathrm{Re}(\psi)) f(1+\mathrm{Re}(\psi))$. That is, Rao's condition does not hold for the subordinators constructed in Subsection 4.4.1.

By Theorem 4.1, we can modify the Lévy measure μ defined in Subsection 4.4.1 by a finite measure and hence obtain a subordinator which does not satisfy Rao's condition and whose Lévy measure μ has a smooth density ρ with respect to the Lebesgue measure on $(0, \infty)$.

2. Besides the index β'' (see (4.3.10)), Blumenthal and Getoor (1961) also introduced the indexes β and σ defined by

$$\beta = \inf\left\{\tau > 0 : \int_{\{|x|<1\}} |x|^\tau \mu(dx) < \infty\right\}$$

and

$$\sigma = \sup\left\{\tau \leq 1 : \int_1^\infty \frac{x^{\tau-1}}{\int_0^\infty (1 - e^{-xy})\mu(dy)} dx < \infty\right\}.$$

From the construction of the subordinators given in Subsection 4.4.1, we obtain by Theorem 6.1 of Blumenthal and Getoor (1961) that

$$\sigma = \beta = \alpha.$$

By (4.4.28) and (4.4.30) (cf. (4.2.3)), we get

$$\beta'' \leq \alpha - \frac{4}{\vartheta}.$$

3. Take $\alpha = 3/4$. For the subordinators constructed in Subsection 4.4.1, we claim that there exists a finite signed measure $d\nu = g_1 dx - g_2 dx$ with $g_1, g_2 \in L^1_+(\mathbf{R}; dx)$ such that

$$\int_{\mathbf{R}} B^{-2}(z)A(z)|\hat{\nu}(z)|^2 dz < \infty \tag{4.4.35}$$

but

$$\limsup_{\lambda\to\infty} \int_{\mathbf{R}} |\hat{\nu}(z)|^2(\lambda + \mathrm{Re}\psi(z))|\lambda + \psi(z)|^{-2} dz = \infty. \tag{4.4.36}$$

Let $\omega \geq 11^{10}$. We define

$$\zeta_\omega(x) := \left\{1 - \frac{1 - 1/(\omega)^{0.1}}{\omega} \cdot |x|\right\}, \quad \text{if } |x| \leq \omega; \quad \frac{1}{|x|^{0.1}}, \text{ otherwise,}$$

and

$$\eta_\omega(x) := \left\{1 - \frac{1 - 1/(\omega)^{0.1}}{\omega} \cdot |x|\right\} \vee 0, \quad x \in \mathbf{R}.$$

By Polya's theorem (cf. Theorem 4.3.1 of Lukacs (1970)), both ζ_ω and η_ω are characteristic functions of absolutely continuous symmetric distributions. Define $\varsigma_\omega := \eta_\omega - \zeta_\omega$. Then, $\varsigma_\omega(x) = 0$ if $|x| \leq \omega$; $\varsigma_\omega(x) = 1/|x|^{0.1}$ if $|x| \geq (1.1)\omega$; and $0 \leq \varsigma_\omega(x) \leq 1/|x|^{0.1}$ otherwise.

For $k \geq 11^{11}$, we define $\xi_k := \varsigma_{\frac{n_k}{2}} - \varsigma_{\frac{2n_k}{1.1}}$. We find that ξ_k is a characteristic function of the difference of two functions $g_1^k, g_2^k \in L^1_+(\mathbf{R}; dx)$ with $\|g_1^k\|_{L^1}, \|g_2^k\|_{L^1} \leq 2$. Define $g_1 := \sum_{k=11^{11}}^{\infty} g_1^k/2^k$, $g_2 := \sum_{k=11^{11}}^{\infty} g_2^k/2^k$ and $d\nu := g_1 dx - g_2 dx$. By applying (4.4.14), (4.4.28), (4.4.32) and the first inequality of (4.4.33) to $B(z)/A(z)$ and applying (4.4.29), (4.4.31) to $B(z)$, we find that there exists a constant $c_5 > 0$ such that

$$\begin{aligned}
\int_{\mathbf{R}} B^{-2}(z)A(z)|\hat{\nu}(z)|^2 dz &= \int_{\mathbf{R}} \frac{1}{\frac{B(z)}{A(z)} \cdot B(z)} |\hat{\nu}(z)|^2 dz \\
&\leq c_5 \sum_{k=11^{11}}^{\infty} \frac{1}{n_k^{\frac{4}{\vartheta} - \frac{1}{22}} \cdot n_k^{2\alpha - 1} \cdot 2^{2k}} \int_{n_k/2}^{2n_k} \frac{1}{z^{0.2}} dz \\
&= c_5 \sum_{k=11^{11}}^{\infty} \frac{1}{n_k^{9/11} \cdot 2^{2k}} \int_{n_k/2}^{2n_k} \frac{1}{z^{0.2}} dz \\
&< \infty.
\end{aligned}$$

However, there exists a constant $c_6 > 0$ such that (cf. (4.2.3 and (4.4.32))

$$\begin{aligned}
\int_{\mathbf{R}} |\hat{\nu}(z)|^2 \frac{n_k^\alpha}{(n_k^\alpha)^2 + (\mathrm{Im}\psi(z))^2} dz &\geq c_6 \frac{1}{n_k^{\frac{3}{4}} \cdot 2^{2k}} \int_{(0.55)n_k}^{\frac{2n_k}{1.1}} \frac{1}{z^{0.2}} dz \\
&\to \infty \ \text{ as } k \to \infty,
\end{aligned}$$

which implies (4.4.36).

It is interesting to compare (4.4.35) and (4.4.36) with the following result, which is a consequence of Theorem 4.8.

Theorem 4.18 *Let X be a Lévy process on $\mathbf{R}^n$ such that all 1-excessive functions are lower semicontinuous. Then (H) holds if and only if*

$$\lim_{\lambda \to \infty} \int_{\mathbf{R}^n} |\hat{\nu}(z)|^2 (\lambda + \mathrm{Re}\psi(z)) |\lambda + \psi(z)|^{-2} dz = 0 \tag{4.4.37}$$

for any finite measure ν of finite 1-energy.

Proof. By Theorem 4.8, Rao (1988) and VI. (4.8) of Blumenthal and Getoor (1968), we need only prove the necessity. Suppose that (H) holds for X. Let ν be a finite measure of finite 1-energy and κ be the standard Gaussian measure on $\mathbf{R}^n$. Then, $\nu+\kappa$ has finite 1-energy, which implies that

$$\int_{\mathbf{R}^n} U^1(\nu+\kappa)d(\nu+\kappa) < \infty. \tag{4.4.38}$$

By (4.4.38), $\kappa(\{x : U^1(\nu+\kappa)(x) = \infty\}) = 0$. Hence $U^1(\nu+\kappa)$ is locally integrable (with respect to the Lebesgue measure dx) by VI. (2.3) of Blumenthal and Getoor (1968). By (H) and VI. (4.9) of Blumenthal and Getoor (1968), we find that $U^1(\nu+\kappa)$ is regular. Therefore, (4.4.37) holds by Theorem 4.8 and the proof is complete. $\square$

So far we have not been able to prove or disprove that (H) holds for the subordinators constructed in Subsection 4.4.1. This example suggests that maybe completely new ideas and methods are needed for resolving Getoor's conjecture.

Chapter 5

Future research

We will consider some problems that are closely related to the thesis.

- For the mixed boundary value problem, we hope to apply some ideas and techniques of Chapter 3 to generalize the recent results of Chen and Zhang (2014) to the case of second order non-symmetric elliptic operators. Our aim is to obtain the probabilistic representation of the solution of the following problem:

$$\begin{cases} Lu = 0 & \text{in } D \\ \frac{\partial u}{\partial \gamma} - < \hat{b}, \mathbf{n} > u = \phi & \text{on } \partial D, \end{cases} \tag{5.0.1}$$

  where the second order elliptic operator L is the same as that we considered in Chapter 3, D is a bounded Lipschitz domain of $\mathbf{R}^n$, $\mathbf{n}$ is the unit inward normal vector on the boundary ∂D, and γ is the conormal vector field on ∂D.

- In Fukushima and Takeda (1984), the authors investigated the results of large deviation by using the symmetric Dirichlet form theory. Since then, Takeda and his students have obtained a lot of nice results on large deviation of time reversal Markov processes. However, to the best of our knowledge, there is no similar result obtained for the non-symmetric Dirichlet forms case. We will consider the large deviation of non-symmetric Dirichlet forms or, more generally, semi-Dirichlet forms by virtue of results of Chapter 2. In particular, we hope to generalize the following result of Takeda (see Takeda (1998)) to the non-symmetric Dirichlet forms case.

Theorem 5.1 (Takeda (1998)) *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form on* $L^2(E; m)$ *for a locally compact separable metric space* E *and a positive Radon measure* m *on* E *with* $\text{supp}[m] = E$*. Suppose* $(X, (P_x)_{x \in E})$ *is the Hunt process associated with* $(\mathcal{E}, \mathcal{F})$*. Let* μ *be a signed smooth measure associated with* $(\mathcal{E}, \mathcal{F})$ *and* A_t^μ *the continuous additive functional corresponding to* μ*. Under some conditions on* $(\mathcal{E}, \mathcal{F})$ *and* μ*,*

$$\lim_{t \to \infty} \frac{1}{t} \log E_x(\exp(-A_t^\mu)) = - \inf_{u \in \mathcal{F}^\mu, ||u||_2 = 1} \left(\mathcal{E}(u, u) + \int_E \tilde{u}\tilde{u} d\mu \right) \tag{5.0.2}$$

for all $x \in E$ *and* $\mathcal{F}^\mu := \{u \in \mathcal{F} : \tilde{u} \in L^2(E; |\mu|)\}$*.*

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